Published by Institute of Physics Publishing for SISSA/ISAS

Received: *June 3, 2003*
Revised: *June 30, 2003*
Accepted: *July 3, 2003*

Traces on orbifolds: anomalies and one-loop amplitudes

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Abstract: In the recent literature one can find calculations of various one-loop amplitudes, like anomalies, tadpoles and vacuum energies, on specific types of orbifolds, like $S^1/\mathbb{Z}_2$. This work aims to give a general description of such one-loop computations for a large class of orbifold models. In order to achieve a high degree of generality, we formulate these calculations as evaluations of traces of operators over orbifold Hilbert spaces. We find that in general the result is expressed as a sum of traces over hyper surfaces with local projections, and the derivatives perpendicular to these hyper surfaces are rescaled. These local projectors naturally takes into account possible non-periodic boundary conditions. As the examples $T^6/\mathbb{Z}_4$ and T^4/D_4 illustrate, the methods can be applied to non-prime as well as non-abelian orbifolds.

Keywords: Field Theories in Higher Dimensions, Anomalies in Field and String Theories, Differential and Algebraic Geometry.

Contents

1. Introduction

In both string theory and field theories of extra dimensions one often considers compactifications on orbifolds. This can either be on their own right, or as approximations of more complicated smooth spaces, like Calabi-Yau or G_2 manifolds, for example. The simplest orbifold $S^1/\mathbb{Z}_2$ has been studied in the context of the eleven dimensional supergravity limit of M-theory [1, 2]. This orbifold also received a lot of attention in five dimensional (GUT) models with (broken) supersymmetric [3]–[15], and has even been studied at two loop [16]. Clearly, since $S^1/\mathbb{Z}_2$ can not be blown up to a smooth manifold, all examples based on this orbifold cannot be related to theories on smooth spaces. On the contrary, four and six dimensional orbifolds, like $T^4/\mathbb{Z}_2$ or $T^6/\mathbb{Z}_3$, can be resolved to give rise to smooth manifolds. All resolutions of four dimensional orbifolds with discrete subgroup of SU(2) result in a (topologically) single manifold, called $K3$. For the six dimensional orbifolds, the possible resolutions are classified by two Hodge numbers. Somewhat surprising, in some sense both string and field theory on orbifolds do not seem to care about whether the orbifolds have resolutions or not: Calculation of physical quantities on either type of orbifolds proceed in an identical fashion. Let us mention a few important computations on orbifolds that are often considered.

Recently there have been substantial investigations to the profile of gauge anomalies on orbifolds. In the context of string theory there have been many investigations to anomalies of zero modes [17, 18]. Horava and Witten were the first to argue, that anomalies on $S^1/\mathbb{Z}_2$ would distribute equally at both fixed points [1, 2], which lead to the discovery of heterotic M-theory. A direct calculation of the shape of the anomaly over $S^1/\mathbb{Z}_2$ has been performed by Arkani-Hamed, Cohen and Georgi with a gauge field and a fermion [19], confirming the Horava-Witten expectation in a five dimensional setting. After that various groups computed the structure of the anomaly on the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$, see [20, 21, 22]. For an investigation of anomalies in a seven dimensional setting compactified on $S^1/\mathbb{Z}_2$ see refs. [23, 24]. Investigations of the shape of anomalies in more than one extra dimensions have also been performed: In ref. [25] the gaugino anomalies in heterotic string theory on $T^6/\mathbb{Z}_3$ with Wilson lines were computed. Anomalies of a six dimensional model on $T^2/\mathbb{Z}_2^3$ can be found in ref. [26]. Very recently, an investigation of anomalies on a more general class of orbifolds has been presented in ref. [27].

Anomaly investigations are very important, as they may provide us with important quantum consistency constraints. However, also other quantum corrections can be vital to gain a more complete understanding of the physics described by a given theory. Let us mention a few of such effects, that have been considered in the context of extra dimensions. To investigate stability issues of a higher dimensional theory from the four dimensional point of view, the effective zero mode potential may proof an important tool [28]–[34]. This potential is obtained by integrating out all Kaluza-Klein modes from the theory. In models where also gravitational effects are considered, one can determine whether the extra dimensions are stabilized and that late time four dimensional cosmology is a possibility, for example.

A different type of stability issue is concerned with the degree of divergence of the fundamental parameters in the classical lagrangian. Most notably, a quadratically divergent Higgs (scalar) mass parameter leads to the well-know hierarchy problem. In supersymmetric theories this divergent scalar mass parameter can be reformulated as a Fayet-Iliopoulos tadpole [35] for an auxiliary field component of a supersymmetric (vector) multiplet. In exact supersymmetry, this term is only renormalized at the one-loop level [36]. This essentially four dimensional discussion has been lifted to five dimensional orbifold theories, showing that these tadpoles can arise on the boundaries of orbifolds, like $S^1/\mathbb{Z}_2$ and $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$ [37, 38, 39, 40, 22], and their consequences have been studied in [41, 42]. Also in higher dimensional cases such tadpoles can be discussed: In six dimensional models this has recently been considered in [43, 44].

Of ten dimensional super Yang-Mills, no auxiliary field formulation exists. However, as has been shown in ref. [45, 46], one can introduce supersymmetric auxiliary field components w.r.t. one of the 4 supersymmetries in four dimensions. On the fixed points of the orbifold $T^6/\mathbb{Z}_3$ these auxiliary fields can develop similar Fayet-Iliopoulos tadpoles for the local anomalous U(1)s. (The tadpoles of zero mode anomalous U(1)s have been discussed in detail in the past in the context of string theory [47]–[51])

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As can been seen from these examples given above, there has not yet been a complete and coherent discussion of all such quantum computations: Only some particular types of orbifolds, and quantum amplitudes have been considered. This article aims to give a unified description of all possible one-loop quantum amplitudes on arbitrary (flat) orbifolds in any given number of dimensions. These orbifolds may be compact or non-compact (like $\mathbb{R}/\mathbb{Z}_2$, for example) or even non-abelian. Taking the orbifold group isomorphic to a finite subgroup of SU(2), orbifolds are obtained that have so-called ADE singularities. In addition, in the compact directions of the orbifold one may introduce Wilson lines or Scherk-Schwarz (super)symmetry breaking [52, 53]. (Discrete Wilson lines were first considered in the context of string theory in [54, 55].)

In order to arrive at a sufficiently general discussion of all these different quantum computations for various orbifold theories, it is convenient to employ a somewhat more abstract description, in terms of Hilbert spaces associated to bundles over orbifolds. Let us recall the connection between these mathematical concepts and the field theory. Any field defines a function of spacetime to a complex vector space. If the spacetime is topologically non-trivial, the field is described by a (local) section of the corresponding complex fiber bundle. The inner product on this fiber can be used to define an inner product on the space of all such sections, turning it into a Hilbert space. One-loop amplitudes can then be formulated as the traces of operators over this Hilbert space. This basic strategy allows us to arrive at an unified description of the computation of all such one-loop quantities on orbifolds.

Paper organization. Section 2 is devoted to provide the mathematical basis for this paper. Subsection 2.1 discusses the geometry of flat orbifolds, in terms of the properties of a torus lattice Γ and an orbifold group G. The later part of that subsection identifies the orbifold fixed points, and describes their properties in some detail. The following subsection describes bundles on orbifolds in terms of homomorphisms of the torus lattice and the orbifold group. The Hilbert spaces associated with these bundles are introduced. Subsection 2.3 describes a useful isomorphism between Hilbert spaces of periodic fields and fields that are periodic up to the orbifold group homomorphism. In addition, a projection operator on the torus Hilbert space is defined to obtain states, that descent down onto the orbifold.

Section 3 is the main part of this work. In subsection 3.1 traces on the various Hilbert spaces are defined. Next, the trace on a general orbifold is evaluated; the details of this calculations have been collected in appendix B. The next two subsections discuss a couple of important applications of the results obtained in this work: the computation of anomalies, tadpoles and vacuum energies on orbifolds.

The material in these two sections has been presented in a rather abstract fashion, therefore it might be helpful to the reader to see how the various concepts can be applied to concrete examples of orbifolds. In section 4 we consider three orbifolds in various dimensions: First, in subsection 4.1 we calculate the trace of a general operator on the well-studied orbifold $S^1/\mathbb{Z}_2$ as familiar illustration. In the following subsection we focus on the orbifold $T^6/\mathbb{Z}_4$ which has an orbifold group that has a subgroup, namely $\mathbb{Z}_2$. In subsection 4.3 we give an example of a non-abelian orbifold T^4/D_4, where D_4 is the dihedral group with eight elements.

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Finally, in section 5 we summarize the main conclusions of this work. In appendix A we expose a couple useful properties of scalar mode functions on tori.

2. Hilbert spaces associated with bundles over orbifolds

In this section we develop some general material to describe field theories on a large class of orbifolds. The discussion here is necessarily a bit abstract and could be some what difficult to follow at first reading. Therefore, we encourage the reader to simultaneously consider the examples provided in section 4. These examples have been chosen both to be familiar to the reader (the $S^1/\mathbb{Z}_2$ example) as well as to illustrate all subtleties that arise for non-prime and non-abelian orbifolds.

2.1 Orbifold geometry

We begin by defining the orbifold $\mathbb{T}/G$ which is sufficiently general, to allow us to describe the various types of orbifolds referred to in the introduction. The space $\mathbb{T} = \mathbb{R}^{1,d-1}/\Gamma$ is defined as a quotient, using an integral lattice $\Gamma \cong \mathbb{Z}^n$ in d-dimensional Minkowski space $\mathbb{R}^{1,d-1}$. In the context of compactification one considers the case where Γ act on the spatial part of the Minkowski space, then we obtain $d-n$ dimensional Minkowski space times an n dimensional torus: $\mathbb{T} = \mathbb{R}^{1,d-n-1} \otimes T^n$ with $n < d$. If Γ has one basis vector in the time direction, we can use $\mathbb{T}$ to perform finite temperature calculations. The inner product $x^T \eta\, y$ for $x, y \in \mathbb{R}^{1,d-1}$ is defined in terms of the diagonal matrix $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$ and x^T denotes the transposed of the vector x. Let G be a finite group, that acts on $\mathbb{R}^{1,d-1}$, and preserves the minkowskian inner product and the orientation of $\mathbb{R}^{1,d-1}$; we take $G \subset SO(1, d-1)$. In addition, it has to be compatible with the lattice Γ that defines the torus, i.e. for all $g \in G$ we have $g\Gamma = \Gamma$. In general the group G does not act freely on $\mathbb{T}$; there may be subspaces of $\mathbb{T}$ which are fixed elements of the orbifold group. It should be stressed that elements of G may act in both the compact and non-compact directions of $\mathbb{T}$. To mention a few concrete examples of orbifolds that can be treated using this formalism: $\mathbb{R}^{1,3} \otimes \mathbb{R}/\mathbb{Z}_2$, $\mathbb{R}^{1,3} \otimes (T^2 \otimes \mathbb{C})/\mathbb{Z}_4$ and the examples considered in section 4.

Each element $g \in G$ generates an abelian subgroup $\langle g \rangle = \{g^k \,|\, 0 \le k < |g|\}$, which is isomorphic to $\mathbb{Z}_{|g|}$. Here $|g|$ is the order of g, i.e. the smallest number n such that $g^n = 1$. We define an operator $P_g^{||}$ for each $g \in G$, that projects on the subspace $\mathbb{T}_g^{||}$ on which g acts as the identity. This definition, and many others that follow below, are all assumed to be defined on the covering space $\mathbb{R}^{1,d-1}$. After that, it is not difficult to apply these definitions to $\mathbb{T}$ as well. This projection operator has the following properties

$$P_g^{||} = \frac{1}{|g|} \sum_{k=1}^{|g|} g^k \,, \qquad P_g^{||} = (P_g^{||})^2 = g\, P_g^{||} = P_g^{||}\, g = P_{g^{-1}}^{||} \,. \tag{2.1}$$

In addition, we have that $= P_{g^p}^{||} = P_g^{||}$ if p and $|g|$ are relatively prime, i.e. $\gcd(p, |g|) = 1$, since then g^p and g generate the same subgroup. Moreover, we define $P_g^{\perp} = 1 - P_g^{||}$. Using these projectors, we can decompose the space $\mathbb{T}$ as

$$\mathbb{T} = \mathbb{T}_g^{||} \otimes \mathbb{T}_g^{\perp} \,, \qquad \mathbb{T}_g^{||} = P_g^{||}\, \mathbb{T} \,, \qquad \mathbb{T}_g^{\perp} = P_g^{\perp}\, \mathbb{T} \,. \tag{2.2}$$

The traces $d_g^{||} = \text{tr}\ P_g^{||}$ and $d_g^{\perp} = \text{tr}\ P_g^{\perp} = d - d_g^{||}$ give the real dimensionality of $\mathbb{T}_g^{||}$ and $\mathbb{T}_g^{\perp}$, respectively. By construction $\langle g \rangle$ acts trivially on $\mathbb{T}_g^{||}$, while g does not act freely on $\mathbb{T}_g^{\perp}$. We define the fixed space of g in $\mathbb{T}$ as

$$\mathbb{T}_g^{\text{fix}} = \{ x \in \mathbb{T} \mid (1-g)x \in \Gamma \} \,. \tag{2.3}$$

The codimension of the space fixed by g is equal to $d_g^{\perp}$; i.e. $\mathbb{T}_g^{\text{fix}} \cap \mathbb{T}_g^{\perp}$ gives exactly the set of fixed points of g in $\mathbb{T}_g^{\perp}$.

In order to obtain unique definitions of these fixed points in $\mathbb{T}_g^{\perp}$ in the covering space $\mathbb{R}^{1,d-1}$, we make the following definition of the fundamental domain of the torus $\mathbb{T}$ in this covering space: $\mathcal{F}(\mathbb{T}) = \{ x^i e_i + y^a \tilde{e}_a \mid 0 \leq x^i < 1,\ y^a \in \mathbb{R} \}$ where e^i are n basis vectors of Γ and $\tilde{e}_a$ are the additional basis vectors pointing in the non-compact directions to form a basis for $\mathbb{R}^{1,d-1}$. The fixed points $\mathfrak{Z}_g^s \in \mathcal{F}(\mathbb{T}_g^{\perp})$ of $\mathbb{T}_g^{\perp}/\langle g \rangle$ are chosen such that they have smallest distance to the origin in the fundamental domain $\mathcal{F}(\mathbb{T})$. This description of the fixed points

$$g\, \mathfrak{Z}_g^s - \mathfrak{Z}_g^s = v_g^s \in \Gamma \tag{2.4}$$

defines the vectors v_g^s uniquely w.r.t. this fundamental domain. The fixed space of g can be decomposed as

$$\mathbb{T}_g^{\text{fix}} = \sum_s \mathbb{T}_{g\,s}^{\text{fix}} + \Gamma \,, \qquad \mathbb{T}_{g\,s}^{\text{fix}} = \mathfrak{Z}_g^s + \mathbb{T}_g^{||} \cong \mathbb{T}_g^{||} \,. \tag{2.5}$$

We define the delta function on $\mathbb{T}$ as $\delta_{\mathbb{T}}(x) = \delta(x - \Gamma)$, where $\delta(x)$ is the delta function on $\mathbb{R}^{1,d-1}$. Because of the direct product structure (2.2) of $\mathbb{T}$ w.r.t. $g \in G$, the torus delta function factorizes as $\delta_{\mathbb{T}}(x) = \delta_g^{||}(x)\, \delta_g^{\perp}(x)$. Here $\delta_g^{||,\perp}(x) = \delta_g^{||,\perp}(P_g^{||,\perp} x)$ denote the delta function on $\mathbb{T}_g^{||,\perp}$. (We assume for any other object with ${}_g^{||,\perp}$ that its coordinate dependence has been restricted to $\mathbb{T}_g^{||,\perp}$.) The delta function $\delta_g^{\perp}$ has the important property:

$$\delta_g^{\perp}(x - g\,x) = \frac{1}{|\det_g^{\perp}(1-g)|} \sum_s \delta_g^{\perp}(x - \mathfrak{Z}_g^s) \,. \tag{2.6}$$

Here $\det{}_g^{\perp}(1-g)$ denotes the determinant on $\mathbb{T}_g^{\perp}$, and for that reason it is non-vanishing. In particular, we set it equal to unity for $g = 1$.

For any function F we define the integral over $\mathbb{T}_g^{\text{fix}}$ as

$$\int_{\mathbb{T}_g^{\text{fix}}} \mathrm{d}x\, F(x) = \sum_s \int_{\mathbb{T}} \mathrm{d}x\, F(x_g^{||} + \mathfrak{Z}_g^s)\, \delta_g^{\perp}(x - \mathfrak{Z}_g^s) \,. \tag{2.7}$$

Many of the properties listed above only depend on the conjugacy class $(g) = \{h^{-1}gh | h \in G\}$ in the group G, but not on the particular element g in the conjugacy class. To show this, we observe that under conjugation we have the following identities

$$P_{hgh^{-1}}^{||,\perp} = h\, P_g^{||,\perp}\, h^{-1} \,, \qquad \mathbb{T}_{hgh^{-1}}^{||,\perp} = h\, \mathbb{T}_g^{||,\perp} \,, \qquad \mathbb{T}_{hgh^{-1}}^{\text{fix}} = h\, \mathbb{T}_g^{\text{fix}} \,, \qquad d_{hgh^{-1}}^{||,\perp} = d_g^{||,\perp} \,,$$
$$\mathfrak{Z}_{hgh^{-1}}^s = h\, \mathfrak{Z}_g^s \,, \qquad v_{hgh^{-1}}^s = h\, v_g^s \,, \qquad \det^{\perp}_{hgh^{-1}}(1 - hgh^{-1}) = \det^{\perp}_g(1-g) \,, \tag{2.8}$$

using that $h\,\mathbb{T} = \mathbb{T}$, $h\,\Gamma = \Gamma$; and similarly for the dual vector spaces $\mathbb{T}_g^{||,\perp *}$, defined in (A.1). In each conjugacy class (g_r) we choose a representative g_r. Each element $h \in (g_r)$ can be written as $h = kg_rk^{-1}$, with k an element of the coset $G/C(g_r)$, with the centralizer $C(g) = \{h \in G \,|\, hgh^{-1} = g\}$.

With these definitions and relations we can investigate the structure of the fixed sets $\mathbb{T}_g^{\text{fix}}$ orbifolded by the action of G. First of all, because of the conjugation property (2.8) of $\mathbb{T}_g^{\text{fix}}$ under the action of $h \in G$ it follows, that all fixed spaces $\mathbb{T}_k^{\text{fix}}$ corresponding to one conjugacy class, i.e. $k \in (g_r)$ are identified. This leads to the identity

$$\Big(\bigcup_{k \in (g_r)} \mathbb{T}_k^{\text{fix}} \Big) \Big/ G = \mathbb{T}_{g_r}^{\text{fix}} / C(g_r)\,. \tag{2.9}$$

Of course, as soon as the group G is abelian, each group element is its own conjugacy class and $C(g) = G$, so that the relation above becomes trivial. Even though the centralizer $C(g)$ maps $\mathbb{T}_g^{\text{fix}}$ to itself, it does not necessarily fix all points in $\mathbb{T}_g^{\text{fix}}$. In fact, $C(g)$ gives rise to an equivalence relation between the labels of the fixed points: $s \sim s'$ if there is a $h \in C(g)$ such that $h\mathfrak{z}_g^s = \mathfrak{z}_g^{s'} + \Gamma$. Upon orbifolding, this leads to the identification of $\mathbb{T}_{g\,s}^{\text{fix}}$ and $\mathbb{T}_{g\,s'}^{\text{fix}}$. We denote the representatives of the corresponding equivalence classes by s_r, and let $G_g^s \subset C(g)$ be the subset, which fixes the fixed point $\mathfrak{z}_g^s$ of g. One should realize that this identification of $\mathbb{T}_{g\,s}^{\text{fix}}$ and $\mathbb{T}_{g\,s'}^{\text{fix}}$ might be non-trivial. To find out how the identifications are preformed, we recall that the structure of the fixed space $\mathbb{T}_{g\,s}^{\text{fix}}$ is given by (2.5). Therefore, the identifications are determined by the mapping

$$p_g^{||}:\; C(g) \to \text{Diff}(\mathbb{T}_g^{\text{fix}})\,; \qquad p_g^{||}(h) = P_g^{||}\, h\, P_g^{||} = h\, P_g^{||} = P_g^{||}\, h\,, \tag{2.10}$$

since $h \in C(g)$. The kernel of this group homomorphism is denoted by $\text{Ker}_g^{||}$. Upon the identifications discussed above, we find that

$$\mathbb{T}_g^{\text{fix}} / C(g) = \sum_{s_r} \mathbb{T}_{g\,s_r}^{\text{fix}} / G_g^{s_r}\,, \tag{2.11}$$

where the sum is over the representatives of the equivalence classes of fixed points of g. This means, that $\mathbb{T}_{g\,s_r}^{\text{fix}} / G_g^{s_r}$ represents an orbifold of dimension $d_g^{||}$, in general. However, part of the group $G_g^{s_r}$ may act trivially on $\mathbb{T}_{g\,s_r}^{\text{fix}}$. Since by definition $G_g^{s_r}$ fixes $\mathfrak{z}_g^{s_r}$, it follows that the kernel

$$\text{Ker}_{g\,s_r}^{||} = \Big\{ h \in G_g^{s_r} \,|\, p_g^{||}(h) = P_g^{||} \Big\} = \text{Ker}_g^{||} \bigcap G_g^{s_r}\,, \tag{2.12}$$

characterizes the trivial part of $G_g^{s_r}$.Taking all this into account we finally obtain

$$\Big(\bigcup_{h \in (g_r)} \mathbb{T}_h^{\text{fix}} \Big) \Big/ G = \sum_{s_r} \mathbb{T}_{g_r\,s_r}^{\text{fix}} / H_{g_r}^{s_r} \qquad H_g^{s_r} = G_g^{s_r} / \text{Ker}_{g\,s_r}^{||}\,. \tag{2.13}$$

In this formula there is no residual trival action of the orbifold groups.

2.2 Bundles of fibers with inner products

Next we give a short description of bundles over the orbifold $\mathbb{T}/G$. To gain an overview of the possible bundles over this orbifold, it is convenient to start with bundles over the covering space $\mathbb{R}^{1,d-1}$. Any bundle over $\mathbb{R}^{1,d-1}$ with fiber $\mathbb{F}$ is trivial, i.e. it is direct product $\mathbb{R}^{1,d-1} \otimes \mathbb{F}$, because the base space $\mathbb{R}^{1,d-1}$ is topologically trivial. We assume, that the fiber $\mathbb{F}$ is a complex vector space of complex dimension N equipped with an inner product $\psi^\dagger \, \rho \, \phi$ defined for all $\psi, \phi \in \mathbb{F}$. Here ρ is an hermitean matrix. It can be simply the identity, or the minkowskian metric η for the tangent bundle, or γ_0 for fermions, to name a couple of important examples. By enforcing suitable reality conditions, the complex fiber $\mathbb{F}$ can be turned into a real vector space. On the torus $\mathbb{T} = \mathbb{R}^{1,d-1}/\Gamma$ not all bundles are trivial: since in general $(\mathbb{R}^{1,d-1} \otimes \mathbb{F})/\Gamma$ is not equal to $\mathbb{T} \otimes \mathbb{F}$. However, the former is merely notation, unless we specify how Γ acts on the fiber $\mathbb{F}$. If it acts as the identity, sections are simply periodic functions $\psi : \ \mathbb{T} \to \mathbb{F}$, and the bundle is trivial: $\mathbb{T} \otimes \mathbb{F}$. The space $\mathcal{H}_\mathbb{T}$ of these sections is a Hilbert space, w.r.t. to the natural inner product

$$\langle \psi \,|\, \phi \rangle_\mathbb{T} = \int_\mathbb{T} \mathrm{d}x \, \psi(x)^\dagger \, \rho \, \phi(x) \,, \quad \forall \psi, \phi \in \mathcal{H}_\mathbb{T} \,. \tag{2.14}$$

We can obtain non-trivial bundles by requiring that the section ψ is only periodic

$$\psi \in \mathcal{H}_{\mathbb{T},T} : \quad \psi(x+v) = T_v \, \psi(x) \,, \qquad T_v \in \mathrm{U(N;}\rho) \,, \qquad v \in \Gamma \,, \tag{2.15}$$

up to an $\mathrm{U(N;}\rho) = \{ S : \mathbb{F} \to \mathbb{F} \,|\, S^\dagger \rho S = \rho \}$ transformation. This requirement ensures that the inner product is preserved by these transformations. The inner product $\langle \psi \,|\, \phi \rangle_{\mathbb{T},T}$ on the corresponding Hilbert space $\mathcal{H}_{\mathbb{T},T}$ is defined in the same way as (2.14). The properties of T_v will be developed below.

To define bundles on the orbifold $\mathbb{T}/G$ we need to specify how the group G acts on the fiber, again using sections, we have

$$\psi \in \mathcal{H}_{\mathbb{T},T,R} : \ \psi(g\, x) = R_g \, \psi(x) \,, \qquad R_g \in \mathrm{U(N;}\rho) \,, \qquad g \in G \,. \tag{2.16}$$

The volume of the orbifold $\mathbb{T}/G$ is a factor $1/|G|$ smaller than that of the torus $\mathbb{T}$ (one might say that the torus is made up of $|G|$ copies of $\mathbb{T}/G$). Therefore, the inner product on the associated orbifold Hilbert space $\mathcal{H}_{\mathbb{T}/G,T,R}$ reads

$$\langle \psi \,|\, \phi \rangle_{\mathbb{T}/G,T,R} = \frac{1}{|G|} \, \langle \psi \,|\, \phi \rangle_\mathbb{T} \,. \tag{2.17}$$

Here we have employed the natural isomorphism between the Hilbert spaces $\mathcal{H}_{\mathbb{T},T,R} \to \mathcal{H}_{\mathbb{T}/G,T,R}$. (We return to this isomorphism in the next subsection.) This inner product does not depend on the fundamental domains chosen, since they are invariant under the identifications induced by Γ and G because of the unitarity of T_v and R_g.

By repeated application of the boundary conditions (2.15) and (2.16) it follows that, T_v and R_g define group homomorphisms

$$T_{v+w} = T_v \, T_w \,, \qquad R_{g\,h} = R_g \, R_h \,, \qquad T_0 = R_1 = 1 \,, \tag{2.18}$$

of Γ and G, respectively. Therefore, in particular we have that $(R_g)^{|g|} = 1$. Since $\psi(x + v + w - v - w) = \psi(x)$ and $\psi(g(g^{-1}x + v) - g\,v) = \psi(x)$, consistency of these boundary conditions requires that

$$[T_v, T_w] = 0\,, \qquad T_{g\,v} = R_g\, T_v\, R_g{}^{-1}\,. \tag{2.19}$$

(Hence, if R_g and T_v commute, one obtains $T_{g\,v} = T_v$.) Such consistency condition have been discussed in ref. [56] in the context of five dimensional models. Moreover, we have for any element $g \in G$ that

$$v \in \Gamma_g^{\perp} \quad \Rightarrow \quad (T_v R_g)^{|g|} = 1\,, \tag{2.20}$$

with $\Gamma_g^{\perp} = P_g^{\perp}\Gamma$ since $\psi(x + \sum g^k v) = \psi(x)$. This implies that T_v is quantized. (If R_g and T_v commute, we find that this reduces to $T_v{}^{|g|} = 1$. If the order $|g|$ is non-prime additional restrictions may arise, see the examples in sections 4.2 and 4.3.) When the fiber contains a Lie algebra, T_v is sometimes referred to as a discrete Wilson line.

Finally, we investigate what happens in the bundle at the fixed spaces of the orbifold. For a point x in the fixed space $\mathbb{T}^{\text{fix}}_{g\,s}$ we find that $R_g\,\psi(x) = \psi(g\,x) = \psi(x + v_g^s) = T_{v_g^s}\,\psi(x)$. Therefore, at $\mathbb{T}^{\text{fix}}_{g\,s}$ only those states exist, that survive the projection

$$R_g^s\,\psi(x) = \psi(x)\,, \qquad R_g^s = T_{v_g^s}{}^{-1} R_g\,, \qquad x \in \mathbb{T}^{\text{fix}}_{g\,s}\,. \tag{2.21}$$

2.3 Hilbert space isomorphisms and operators

To facilitate the computation of traces over orbifold Hilbert spaces later, it is useful to identify isomorphisms between some relevant Hilbert spaces. On the level of the torus $\mathbb{T}$ we have the isomorphism

$$\mathcal{T}:\ \mathcal{H}_{\mathbb{T}} \to \mathcal{H}_{\mathbb{T},T}\,; \qquad (\mathcal{T}\psi)(x) = \mathcal{T}(x)\psi(x)\,. \tag{2.22}$$

Here the function $\mathcal{T}(x)$ is obtained as follows: Since all T_v commute (2.19) for all $v \in \Gamma$, it follows that there are commuting generators H_I such that $T_v = \exp(i\,a^T \eta\, v)$ where $a^\mu = \sum_I a^{\mu\,I} H_I$ and η represent the minkowskian metric. Using this, we can easily construct a function $\mathcal{T}(x)$ which is Γ-periodic up to T transformations:

$$\mathcal{T}(x) = e^{i\,a^T \eta\, x}\,, \qquad \mathcal{T}(0) = 1\,, \qquad \forall v \in \Gamma:\ \mathcal{T}(x + v) = T_v\,\mathcal{T}(x)\,. \tag{2.23}$$

In other words, the function $\mathcal{T}(x)$ can be used to turn periodic functions into functions periodic up to T_v transformations, and vice versa. The projection equation (2.21) at $\mathbb{T}^{\text{fix}}_g$ can be represented as

$$\mathcal{R}_g(x)\,\psi(x) = \psi(x),\ \forall x \in \mathbb{T}^{\text{fix}}_g\,; \quad \text{with} \quad \mathcal{R}_g(x) = \mathcal{T}_g^{\perp}(x)\,\mathcal{T}_g^{\perp}(g\,x)^{-1}\,R_g\,. \tag{2.24}$$

To see that this is equivalent with the condition given above, notice that for $x \in \mathbb{T}^{\text{fix}}_{g\,s}$: $\mathcal{R}_g(x) = \mathcal{R}_g(\mathfrak{z}_g^s) = \mathcal{T}_g^{\perp}(\mathfrak{z}_g^s)\,\mathcal{T}_g^{\perp}(g\,\mathfrak{z}_g^s)^{-1}\,R_g = R_g^s$. This notation can be used to avoid having to specify with which subspace $\mathbb{T}^{\text{fix}}_{g\,s}$ of $\mathbb{T}^{\text{fix}}_g$ one is concerned with.

In (2.17) the inner product on the orbifold Hilbert space $\mathcal{H}_{\mathbb{T}/G,T,R}$ had been expressed in terms of the inner product on the torus Hilbert space. However, not all states in the torus Hilbert space $\mathcal{H}_{\mathbb{T},T,R}$ descent down to states on the orbifold. To obtain the states which do, we define the orbifold projection operator

$$\mathcal{P}_R : \ \mathcal{H}_{\mathbb{T},T} \to \mathcal{H}_{\mathbb{T},T}; \qquad (\mathcal{P}_R\psi)(x) = \frac{1}{|G|}\sum_{g\in G} R_g\,\psi(g^{-1}x)\,. \tag{2.25}$$

For $\psi \in \mathcal{H}_{\mathbb{T},T}$, i.e. $\psi(x+v) = T_v\psi(x)$ it follows indeed that

$$(\mathcal{P}_R\psi)(h\,x) = R_h\,(\mathcal{P}_R\psi)(x)\,, \qquad (\mathcal{P}_R\psi)(x+v) = T_v\,(\mathcal{P}_R\psi)(x)\,, \tag{2.26}$$

because any $h \in G$ defines a group automorphism by $h(g) = hg$, R is a homomorphism (2.18), and formula (2.19) for $T_{g^{-1}v}$. Moreover, the operator $\mathcal{P}_R$ satisfies the usual properties of a projection operator

$$\mathcal{P}_R{}^2 = \mathcal{P}_R{}^\dagger = \mathcal{P}_R\,. \tag{2.27}$$

These properties follow, using again that any $h \in G$ defines an automorphism on G. In addition, the Hermiticity property required a change of variables in the integral over $\mathbb{T}$ within the Hilbert space inner product. The natural isomorphism $\mathcal{H}_{\mathbb{T},T,R} \to \mathcal{H}_{\mathbb{T}/G,T,R}$ allows us to perform all calculations in the orbifold Hilbert space in the Hilbert space of torus states with appropriate boundary conditions instead. In fact, we may simply work with the states of the Hilbert space $\mathcal{H}_{\mathbb{T},T}$, and use the operator $\mathcal{P}_R$ to project on to $\mathcal{H}_{\mathbb{T},T,R}$. Since the former has a much simpler structure, this proves to be a very convenient strategy.

Finally, we discuss some properties of general operators on $\mathcal{H}_{\mathbb{T},T}$ and $\mathcal{H}_{\mathbb{T}/G,T,R}$, which turn out to be of central importance in the subsequent discussion. Let $\mathcal{O}$ be an operator on $\mathcal{H}_{\mathbb{T},T}$. Its expectation value in the state $|\psi\rangle$ is denoted by $\langle\psi\,|\,\mathcal{O}\,|\,\psi\rangle_{\mathbb{T},T}$. A coordinate space representation of this operator is written as $\mathcal{O}(x,\partial_x)$. To avoid that the derivatives act on the operator itself, we assume that the operators x and ∂_x are ordered such that the operators ∂_x stand on the right:

$$\mathcal{O}(x,\partial_x) = \sum_{\vec\mu} \mathcal{O}^{\vec\mu}(x)\,\partial_{\vec\mu} = \sum_k \sum_{\mu_1\ldots\mu_k} \mathcal{O}^{\mu_1\ldots\mu_k}(x)\,\partial_{\mu_1}\ldots\partial_{\mu_k}, \tag{2.28}$$

where the sum is over all multi-indices $\vec\mu = (\mu_1,\ldots,\mu_k)$ and $\partial_{\vec\mu} = \partial_{\mu_1}\ldots\partial_{\mu_k}$ for all integral $k \geq 0$, and $\mu_i = 0,\ldots,d-1$. In general the different $\mathcal{O}^{\vec\mu}(x)$ do not mutually commute, and may also not commute with R_g and T_v.

Since the orbifold Hilbert space can be viewed as a subset of the torus Hilbert space, also not all operators on the torus Hilbert space make sense on the orbifold Hilbert space. Those operators that do, are called compatible with the orbifold boundary conditions. To investigate what the defining property of such operators is, let $\mathcal{O}\,:\ \mathcal{H}_{\mathbb{T},T} \to \mathcal{H}_{\mathbb{T},T}$ and $\psi \in \mathcal{H}_{\mathbb{T},T,R}$ be an orbifold state, i.e. $\psi(g\,x) = R_g\,\psi(x)$. If $\mathcal{O}$ is an orbifold compatible operator, then also $\mathcal{O}\,\psi$ should be an element of $\mathcal{H}_{\mathbb{T},T,R}$. By applying the orbifold boundary conditions to $\mathcal{O}\,\psi$, we find that

$$\mathcal{O}(g\,x, \partial_x\,g^{-1}) = R_g\,\mathcal{O}(x,\partial_x)\,R_g{}^{-1}\,. \tag{2.29}$$

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From any operator $\mathcal{O}: \mathcal{H}_{\mathbb{T},T} \to \mathcal{H}_{\mathbb{T},T}$ an orbifold compatible operator $\mathcal{O}_R$ can be obtained by

$$\mathcal{O}_R(x, \partial_x) = \frac{1}{|G|} \sum_{h \in G} {R_h}^{-1} \mathcal{O}(h\,x, \partial_x\, h^{-1})\, R_h\,. \tag{2.30}$$

In fact, this operator will arise naturally when we compute traces on orbifolds, see section 3. If $\mathcal{O}$ is an orbifold compatible operator, it follows trivially that $\mathcal{O}_R = \mathcal{O}$.

3. Orbifold Hilbert space traces and their applications

In this section we expose our main result concerning the traces on orbifolds: It describes how the trace of an operator on a Hilbert space of a non-trivial orbifold bundle, can be reduced to a sum over traces over Hilbert spaces associated to trivial ones. The various technicalities of the computations are collected in appendix B. After this general exposition of the computation of Hilbert space traces on orbifolds, we discuss a few possible applications of this general method: In subsections 3.2 and 3.3 we consider anomalies, and tadpoles on orbifolds, respectively.

3.1 Tracing over the orbifold Hilbert space

A orthonormal basis $\{|\phi_\sigma(q)\rangle\}$ for the Hilbert space $\mathcal{H}_{\mathbb{T}}$ has been constructed in appendix A, see eq. (A.8). Here σ labels the basis vectors ϵ_σ of the fiber $\mathbb{F}$, and $q \in \mathbb{T}^*$ takes its values in the vector space dual to $\mathbb{T}$. It should be noted that none of our results depend on this explicit basis. We define the trace $\mathrm{Tr}_{\mathbb{T}}$ of an operator $\mathcal{O}$ on the Hilbert space $\mathcal{H}_{\mathbb{T}}$ associated to a trivial bundle over a torus $\mathbb{T}$ as

$$\mathrm{Tr}_{\mathbb{T}}\,[\mathcal{O}] = \sum_\sigma \int_{\mathbb{T}^*} \mathrm{d}q\, \langle \phi_\sigma(q)\,|\, \mathcal{O}\,|\,\phi_\sigma(q)\rangle_{\mathbb{T}} = \sum_\sigma \int_{\mathbb{T}\otimes\mathbb{T}^*} \mathrm{d}x\, \mathrm{d}q\;\, \phi_\sigma(x;q)^\dagger\, \mathcal{O}(x,\partial_x)\, \phi_\sigma(x;q)\,. \tag{3.1}$$

For fermionic fields there will be an additional minus sign; we will always write it explicitly. In general such traces are defined only formally, as they may not converge and regularization is required. In the following we assume that either the operator $\mathcal{O}$ is a bounded operator, or that it has been regularized such that it has become a bounded operator. From this trace on a trivial bundle over the torus, we can define the trace on a torus bundle with periodicity conditions (2.15) defined by the homomorphism T. Because of the isomorphism (2.22) between $\mathcal{H}_{\mathbb{T}}$ and $\mathcal{H}_{\mathbb{T},T}$, the trace on $\mathcal{H}_{\mathbb{T},T}$ reads:

$$\mathrm{Tr}_{\mathbb{T},T}\,[\mathcal{O}] = \mathrm{Tr}_{\mathbb{T}}\left[\mathcal{T}^\dagger\, \mathcal{O}\, \mathcal{T}\right]. \tag{3.2}$$

In turn, we can use this to define the trace $\mathrm{Tr}_{\mathbb{T}/G,T,R}$ on the non-trivial orbifold bundle defined by the orbifold twist R, exploiting the defining isomorphism between $\mathcal{H}_{\mathbb{T}/G,T,R}$ and $\mathcal{H}_{\mathbb{T},T,R}$. Let $\mathcal{O}$ be an compatible operator (2.29) on $\mathcal{H}_{\mathbb{T},T,R}$, so that it can be restricted to the orbifold. We can compute the trace of $\mathcal{O}$ on $\mathcal{H}_{\mathbb{T}/G,T,R}$ as

$$\mathrm{Tr}_{\mathbb{T}/G,T,R}\,[\mathcal{O}] = \frac{1}{|G|}\,\mathrm{Tr}_{\mathbb{T},T,R}\,[\mathcal{O}] = \frac{1}{|G|}\,\mathrm{Tr}_{\mathbb{T},T}\left[{\mathcal{P}_R}^\dagger\, \mathcal{O}\, \mathcal{P}_R\right] = \frac{1}{|G|}\,\mathrm{Tr}_{\mathbb{T}}\left[\mathcal{T}^\dagger {\mathcal{P}_R}^\dagger\, \mathcal{O}\, \mathcal{P}_R \mathcal{T}\right], \tag{3.3}$$

with the help of the orbifold projector $\mathcal{P}_R$, defined in (2.25). This trace can be evaluated by using the definition of the traces of the trivial bundle on the torus (3.1) and by substituting the definition of the orbifold projector (2.25).

The details of this calculation are rather lengthy, and are therefore given in appendix B; we only give the main results here. We also give some intermediate results, which are slightly more general than required by the calculation at hand: they often apply to any operator on $\mathcal{H}_{\mathbb{T},T}$, not only the orbifold compatible ones. First of all, for any operator on $\mathcal{H}_{\mathbb{T},T}$ the relation

$$\mathrm{Tr}_{\mathbb{T},T,R}[\mathcal{O}] = \mathrm{Tr}_{\mathbb{T},T}\left[\mathcal{P}_R{}^{\dagger}\, \mathcal{O}_R\right] \tag{3.4}$$

holds, where $\mathcal{O}_R$ is the orbifold compatible operator defined in (2.30). Also for any operator on $\mathcal{H}_{\mathbb{T},T}$ we find that

$$\mathrm{Tr}_{\mathbb{T},T}\left[\mathcal{P}_R{}^{\dagger}\mathcal{O}\right] = \frac{1}{|G|}\sum_{g\in G}\frac{1}{|\det{}^{\perp}_{g}(1-g)|}\,\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{g},T^{\parallel}_{g}}\left[\mathcal{R}_g\,\mathcal{O}_g\right], \tag{3.5}$$

using the definition (2.24) and a rescaling of the derivative perpendicular to g:

$$\mathcal{O}_g(x,\partial) = \mathcal{O}\left(x, \partial^{\parallel}_{g} + (\partial(1-g)^{-1})^{\perp}_{g}\right). \tag{3.6}$$

Here the same ordering as in (2.28) has been applied. Notice that for the identity: $\mathcal{O}_1(x,\partial_x) = \mathcal{O}(x,\partial_x)$. In addition, we have used the notation defined in (2.7) and (2.23). It should be noted that this result depends on careful treatment of the properties of the delta function, see the discussion leading to (B.4) of appendix B.

The result of the computation of (3.3) can be obtained by combining (3.4) and (3.5) for an orbifold compatible operator $\mathcal{O}$ on $\mathcal{H}_{\mathbb{T},T}$, for which $\mathcal{O}_R = \mathcal{O}$ see discussion below (2.30). Making use of the discussion of the orbifolding of fixed spaces at the end of subsection 2.1, we can write the trace over the orbifold Hilbert space as

$$\mathrm{Tr}_{\mathbb{T}/G,T,R}[\mathcal{O}] = \frac{1}{|G|}\sum_{g_r}\frac{1}{|\det^{\perp}_{g_r}(1-g_r)|}\sum_{s_r}\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{g_r\,s_r}/H^{s_r}_{g_r},T^{\parallel}_{g_r}}\left[R^{s_r}_{g_r}\,\mathcal{O}_{g_r}\right], \tag{3.7}$$

with $R^{s_r}_{g_r} = T_{v^{s_r}_{g_r}}{}^{-1}R_{g_r}$. The sums here are over the representatives g_r of the conjugacy classes (g_r) of the group G, and the representatives s_r of the equivalence classes of the fixed point labels. Notice that if the homomorphism T is the identity, we would find $R^{s_r}_{g_r} = R_{g_r}$ at all fixed spaces $\mathbb{T}^{\mathrm{fix}}_{g_r\,s_r}/H^{s_r}_{g_r}$. This shows, that the non-periodic torus boundary conditions, encoded by the non-trivial homomorphism T, lead to modifications of the traces at fixed spaces compared to those trace on a pure orbifold.

The remaining traces are rather straightforward since they are over states which are periodic on $\mathbb{T}^{\mathrm{fix}}_{g_r\,s_r}$ up to transformations $T^{\parallel}_{g}$ only. However, there is no orbifold projection in the fiber of the bundle anymore: The orbifold twist is encoded entirely in the matrix $R^{s_r}_{g_r}$. To work these traces out further we need more information concerning the structure of the relevant fibers. However, since at fixed point s_r the matrix $R^{s_r}_{g_r}$ generates an abelian

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group of order $|g_r|$, we may decompose this matrix

$$R_{g_r}^{s_r} = \sum_{k=0}^{|g|-1} e^{2\pi i\, k/|g|}\, P_{g_r}^{s_r\, k} \tag{3.8}$$

in terms of the projection operators $P_{g_r}^{s_r\, k}$ of the possible $|g|$, eigenvalues $\exp(2\pi i\, k/|g|)$, of $R_{g_r}^{s_r}$.

3.2 Anomalies

As mentioned in the introduction of this paper, there has been a lot of attention to the calculation of anomalies on different types of orbifolds. We will not compare our general results with all of them, but rather we focus on the following question: It is well-know that only in even dimensions there are gauge (and gravitational) anomalies. On the other hand it has been argued by various authors that on the orbifold $S^1/\mathbb{Z}_2$ gauge anomalies arise at the fixed points.[1] Only very recently a more general analysis of anomalies on orbifolds has been given in ref. [27]. Here we would like to present a description of gauge anomalies that applies to any orbifold of the type discussed in subsection 2.1. In particular, we allow that the full space and the fixed hyper surfaces can all have either even or odd dimensionality.

We consider a fermion ψ on $\mathbb{T}/G$ coupled to a hermitean (non-)abelian gauge connection A_μ and a spin-connection ω_μ, as this is the typical situation in which anomalies may arise. In general, this fermion may transform under some non-abelian (and chiral for even dimension d) gauge transformation, to which the gauge field A_μ is associated. On the d dimensional space $\mathbb{T}$ with minkowskian signature, its (kinetic) action takes the standard form

$$S = -\int \mathrm{d}^d x\, \bar{\psi} \not{D} \psi\,, \qquad \not{D} = \not{\partial} + i\not{A} + \not{\omega}\,. \tag{3.9}$$

Of course the boundary conditions (2.16) only make sense, if T_v and R_g define symmetries of the action of the fermion. Therefore for all Minkowski indices $\mu = 0, 1, \ldots d-1$ we have that

$$(-\gamma_0 {T_v}^\dagger \gamma_0)\, \gamma^\mu\, T_v = \gamma^\mu\,, \qquad (-\gamma_0 {R_g}^\dagger \gamma_0)\, (g^{-1})^\mu{}_\nu\, \gamma^\nu\, R_g = \gamma^\mu\,. \tag{3.10}$$

The gauge and spin-connections contained in the Dirac operator have to satisfy corresponding boundary conditions for consistency; they read

$$\begin{aligned} (-\gamma_0 {T_v}^\dagger \gamma_0)\, \gamma^\mu\, A_\mu(x+v)\, T_v &= \gamma^\mu\, A_\mu(x)\,, \\ (-\gamma_0 {R_g}^\dagger \gamma_0)\, (g^{-1})^\mu{}_\nu\, \gamma^\nu\, A_\mu(gx)\, R_g &= \gamma^\mu\, A_\mu(x)\,, \\ (-\gamma_0 {T_v}^\dagger \gamma_0)\, \gamma^\mu\, \omega_\mu(x+v)\, T_v &= \gamma^\mu\, \omega_\mu(x)\,, \\ (-\gamma_0 {R_g}^\dagger \gamma_0)\, (g^{-1})^\mu{}_\nu\, \gamma^\nu\, \omega_\mu(gx)\, R_g &= \gamma^\mu\, \omega_\mu(x)\,. \end{aligned} \tag{3.11}$$

The boundary conditions (2.16) may lead to chiral fermions in complex representations at some of the even dimensional fixed hyper surfaces. The conditions (3.11) ensure that the

[1] By Arkani-Hamed et al. [19] the axial anomaly has been computed from first principles; but not the gauge anomaly. Even though both anomalies are similar in structure, there are also important differences.

Dirac operator $\not{D}$ is an orbifold compatible operator (2.29). It should also be realized that the transformation parameter, $\Lambda^\dagger = \Lambda$, satisfies corresponding boundary conditions to the ones given in (3.11) for the gauge and spin-connections. In addition, there may be chiral and/or Majorana projections that act on the spinors, depending on the dimensionality d.

Before continuing with the discussion of the anomalies on the orbifold, let us mention some well-known results concerning anomalies on a smooth manifold $\mathbb{M}$ of dimension d, i.e. without boundaries and singular hyper surfaces. A constant parameter Λ can be any element of the d dimensional Clifford algebra. However, since we only consider gauge fields and the spin-connection, their parameters are proportional to the identity and the spin generators $\frac{1}{4}[\gamma_\mu, \gamma_\nu]$, respectively. Moreover, if the dimension d of $\mathbb{T}$ is even, we may allow that the (gauge) transformations are different on the left- and right-handed fermionic components. Such transformations can be anomalous because the fermionic path integral measure is not necessarily invariant under the fermionic field redefinition $\psi \to (1+\Lambda)\psi$ and $\bar\psi \to \bar\psi(1-\gamma_0\Lambda^\dagger\gamma_0)$. This may lead to the non-conservation of the gauge current J^μ on the manifold $\mathbb{M}$:

$$\langle \partial_\mu J^\mu \rangle_{\mathbb{M}} = \mathcal{A}_{\mathbb{M}}(\Lambda) = -\,\mathrm{Tr}_{\mathbb{M}}\left[\Lambda\, e^{\not{D}^2/M^2}\right] - \mathrm{Tr}_{\mathbb{M}}\left[-\gamma_0\Lambda\gamma_0\, e^{\not{D}^2/M^2}\right], \tag{3.12}$$

the minus signs arise because the trace is over fermionic degrees of freedom. Here we have introduced the so-called heat kernel regularization following the standard method of Fujikawa [57, 58, 59]; without any regularization the traces of the gauge parameters Λ and $-\gamma_0\Lambda^\dagger\gamma_0$ over the full Hilbert spaces of ψ and $\bar\psi$ are ill-defined. The regulator parameter M is taken to infinity at the end of the anomaly calculation. The result is that only in even dimensions there can be anomalies, and they are generally given by

$$\mathcal{A}_{\mathbb{M}}(\Lambda) = 2\pi \int_{\mathbb{M}} \Omega^1_{2k}(\Lambda; A, F, \omega, R) \tag{3.13}$$

where F and R are the field strengths of A and ω, respectively. The $d = 2k$ form Ω^1_{2k} follows from the solution of the descent equations, that solve the Wess-Zumino consistency conditions [60], from the close and invariant $2k+2$ form Ω_{2k+2} [61, 62]:

$$\begin{aligned} \delta_\Lambda \Omega_{2k+2}(F,R) &= \mathrm{d}\Omega_{2k+2}(F,R) = 0\,, \\ \mathrm{d}\Omega_{2k+1}(A,F,\omega,R) &= \Omega_{2k+2}(F,R)\,, \\ \delta_\Lambda \Omega_{2k+1}(A,F,\omega,R) &= \mathrm{d}\Omega^1_{2k}(\Lambda; A,F,\omega,R)\,. \end{aligned} \tag{3.14}$$

The precise particle content determines what the explicit form of the defining polynomial Ω_{2k+2} is; for a spin 3/2 or spin 1/2 field in representation $\mathbf{R}$ of the gauge group we find

$$\Omega^{3/2}_{2k+2} = \hat{A}_{3/2}(R)\Big|_{2k+2}, \qquad \Omega^{1/2,\mathbf{R}}_{2k+2} = \hat{A}_{1/2}(R)\,\mathrm{tr}_{\mathbf{R}}\, e^{iF/2\pi}\Big|_{2k+2}, \tag{3.15}$$

respectively. Here $\hat{A}_s$ with $s = 1/2, 3/2$ refer to the roof genus, for their definitions see [63]–[66].

Let us now return to the orbifold. Because both the gauge parameter Λ as well as the Dirac operator $\not{D}$ are compatible with the orbifolding, the traces in the definition of the anomaly (3.12) are well-defined. Therefore we may apply (3.7) to obtain

$$\mathcal{A}_{\mathbb{T}/G,T,R} = \frac{-1}{|G|} \sum_{g_r, s_r} \frac{1}{|\det^{\perp}_{g_r}(1-g_r)|} \mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{g_r s_r}/H^{s_r}_{g_r}, T^{\parallel}_{g_r}} \Big[\Big(R^{s_r}_{g_r} \Lambda - \gamma_0 R^{s_r\dagger}_{g_r} \Lambda^\dagger \gamma_0 \Big) e^{\not{D}^2/M^2} \Big]_{g_r} . \tag{3.16}$$

A comment is in order here: In this calculation we have taken ψ and $\bar{\psi}$ to be independent. Therefore, we not only have to take into account how the orbifold operator $\mathcal{R}_g$ acts on ψ, but we also remember that on $\bar{\psi}$ it acts as $-\gamma_0 R^{s_r\dagger}_{g_r} \gamma_0$.

To express this in terms of the anomaly forms Ω_{2k}, we need in general more information concerning the precise structure of the bundle in which the fermions take their value. The method of decomposing the $R^{s_r}_{g_r}$ explained in (3.8) can also be applied here.

3.3 Tadpoles and vacuum energy

As a second example of the possible one-loop quantities one can compute on orbifolds using the general machinery exposed in this paper, we would like to mention the computation of one-loop tadpoles and the vacuum energy. These quantities can often be very significant to understand the physics of the underlying theory. In particular, possible (local) Fayet-Iliopoulos tadpoles in supersymmetric theories in higher dimensions may trigger spontaneous symmetry breaking. This may also lead to localization of charged zero modes [22]. These types of tadpoles in general can introduce quadratic sensitivity to a high scale in supersymmetric theories of extra dimensions [37, 38]. Here, we do not aim at full generality, since the computation of both these quantities on regular spaces are well-known; rather we would like to illustrate how such quantities can be computed on general orbifolds. The methods described here can be used to confirm the recent results obtained in ref. [46], in which tadpoles on the orbifold $T^6/\mathbb{Z}_3$ were computed.

Consider a theory with N real scalars ϕ^α with a potential $V(\phi)$. Let us decompose the scalars in their background values ϕ^α_0 and their quantum fluctuations $\delta\phi^\alpha$. The vacuum energy $V(\phi_0)$ and the one-loop tadpoles $T_\alpha(\phi_0)$ read

$$V(\phi_0) = \frac{1}{2} \mathrm{Tr} \ln\Big[-\Box + M^2(\phi_0) \Big] , \qquad T_\alpha(\phi_0) = \frac{1}{2} \mathrm{Tr} \Big[(M^2)_{,\alpha}(\phi_0) \frac{1}{-\Box + M^2(\phi_0)} \Big] , \tag{3.17}$$

with the euclidean laplacian $\Box = \partial^2_\mu$ and the mass matrix $(M^2)_{\alpha\beta}(\phi_0) = V_{,\alpha\beta}(\phi_0)$. As explained in ref. [67], and references therein, the effective potential can be obtained from the tadpoles by integration over the background fields. The definitions above are ill-defined and require regularization. Various methods, like dimensional, zeta-function, or heat-kernel regularization can be applied to do this; see for example ref. [68] for a discussion how these different schemes are related. Here we restrict ourselves to the zeta-function regularization method, and define

$$I_\zeta(\alpha, M^2; \delta) = \mathrm{Tr} \Big(-\Box + M^2 \Big)^{-\alpha-\delta} , \tag{3.18}$$

with α an arbitrary real constant and the regulator $\delta \in \mathbb{C}$, which should be taken to zero at the end of a physical calculation. The regulated expressions for the tadpoles and the vacuum energy read

$$T_\alpha(\phi_0; \delta) = -\frac{1}{2\delta} \frac{\delta}{\delta \phi_0^\alpha} I_\zeta(0, M^2(\phi_0); \delta)\,, \qquad V(\phi_0; \delta) = -\frac{1}{2\delta} I_\zeta(0, M^2(\phi_0); \delta)\,. \tag{3.19}$$

These quantities can now be computed on the orbifold using eq. (3.7), and expressed as a sum of traces on the different spaces $\mathbb{T}^{\text{fix}}_{g_r\, s_r}/H^{s_r}_{g_r}$.

4. Examples

In this section we give three examples of orbifolds to which the general procedure for computing orbifold traces, developed in the preceding sections, can be applied. The first example we discuss is the well-known orbifold $S^1/\mathbb{Z}_2$. As this is a very simple and well-studied orbifold, our general methods actually provide a bit of overkill in this case. However, because of its familiarity it might help the reader to understand the general features of the procedure advocated in this work. The next example concerns a ten dimensional model, for example a super Yang-Mills theory or supergravity, on the orbifold T^6/Z_4. The third example we consider, is the non-abelian orbifold T^4/D_4 with the D_4 the dihedral group with eight elements.

4.1 5D theories on the orbifold $S^1/\mathbb{Z}_2$

As stated in the introduction of this section, this subsection is meant to illustrate the general machinery presented in the preceding part of this article. Consider five dimensional Minkowski space $\mathbb{R}^{1,4}$, the one dimensional lattice and the group

$$\Gamma = \begin{pmatrix} 0_4 \\ \mathbb{Z} \end{pmatrix} \qquad G = \{\mathbb{1}_5, g\} \cong \mathbb{Z}_2\,, \qquad g = \begin{pmatrix} \mathbb{1}_4 & \\ & -1 \end{pmatrix}, \tag{4.1}$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. As the group G is abelian, it follows that the group elements form the conjugacy classes, and the centralizer is the group itself. The only non-trivial projection operators on $\mathbb{T} = \mathbb{R}^{1,3} \times S^1$ read

$$P_g^{||} = \begin{pmatrix} \mathbb{1}_4 & \\ & 0 \end{pmatrix}, \quad P_g^{\perp} = \begin{pmatrix} 0_4 & \\ & 1 \end{pmatrix} \;\Rightarrow\; \mathbb{T}_g^{||} = \begin{pmatrix} \mathbb{R}^{1,3} \\ 0 \end{pmatrix}, \quad \mathbb{T}_g^{\perp} = \begin{pmatrix} 0_4 \\ S^1 \end{pmatrix}, \qquad s = 0, 1\,. \tag{4.2}$$

In particular we find that $\det^{\perp}_g(1-g) = 2$. The fixed points and fixed sets are easily identified

$$\mathfrak{z}_g^s = \begin{pmatrix} 0_4 \\ \frac{1}{2}\, s \end{pmatrix}, \qquad v_g^s = \begin{pmatrix} 0_4 \\ -s \end{pmatrix}, \qquad \mathbb{T}^{\text{fix}}_{g\,s} = \mathfrak{z}_g^s + \begin{pmatrix} \mathbb{R}^{1,3} \\ 0 \end{pmatrix}. \tag{4.3}$$

The mapping $p_g^{||}$ defined in (2.10) is trivial

$$p_g^{||}(g) = P_g^{||}\,, \qquad \text{Ker}_g^{||} = G\,. \tag{4.4}$$

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For the fixed sets in the orbifold we thus find the following

$$\mathbb{T}_1^{\text{fix}}/G = \mathbb{T}/G = \mathbb{R}^{1,3} \times S^1/\mathbb{Z}_2\,, \qquad \mathbb{T}_g^{\text{fix}}/G = \sum_{s=0,1} \mathbb{T}_{g\,s}^{\text{fix}}\,. \tag{4.5}$$

This concludes the geometrical description of the orbifold.

As we described in subsection 2.2 any field theory on this orbifold is essentially determined by two group homomorphisms T and R of Γ and G, respectively, to the group of diffeomorphisms of the relevant fiber vector space in which the field takes its values:

$$\psi(x+2\pi) = T\,\psi(x)\,, \quad \psi(-x) = R\,\psi(x)\,, \qquad R^2 = (TR)^2 = 1\,. \tag{4.6}$$

In the general trace formula on this orbifold this can be left unspecified, and we obtain:

$$\mathrm{Tr}_{\mathbb{T}/G,T,R}\Big[\mathcal{O}\Big] = \frac{1}{2}\,\mathrm{Tr}_{\mathbb{T}/G,T}\Big[\mathcal{O}\Big] + \frac{1}{2}\cdot\frac{1}{2}\sum_{s=0,1}\mathrm{Tr}_{\mathbb{T}^{\text{fix}}_{g\,s}}\Big[R_g^s\mathcal{O}_g\Big]\,, \tag{4.7}$$

where $\mathcal{O}_g(x^i, x^5; \partial_i, \frac{1}{2}\,\partial_5)$ and $R_g^s = T^{-s}R = RT^s$.

Let us work out this formula for the example of the gauge anomaly of a single fermion ψ on this orbifold coupled to an abelian gauge field A_μ and gauge parameter Λ. Upon requiring that there is a four dimensional gauge field zero mode, the orbifold twist action on these fields is uniquely determined to be

$$\begin{aligned}
\psi(x+2\pi) &= (-)^p\psi(x)\,, & \bar\psi(x+2\pi) &= (-)^p\bar\psi(x)\,,\\
A_i(x+2\pi) &= A_i(x)\,, & A_5(x+2\pi) &= A_5(x)\,,\\
\psi(g\,x) &= \gamma_5\,\psi(x)\,, & \bar\psi(g\,x) &= \bar\psi(x)(-\gamma_5)\,,\\
A_i(g\,x) &= A_i(x)\,, & A_5(g\,x) &= -A_5(x)\,,
\end{aligned} \tag{4.8}$$

here $p = 0, 1$ specifies whether we take periodic ($p = 0$) or anti-periodic ($p = 1$) boundary conditions on the circle. (For a single fermion T and R necessarily commute, hence $T^2 = 1$. Requiring that the fermionic action is invariant then implies that $T = (-)^p$.) In the second relation we have used that $-\gamma_0 {R_g}^\dagger\gamma_0 = -\gamma_0\gamma_5\gamma_0 = -\gamma_5$. In equation (3.16) for the general anomaly the part in between the round brackets becomes for the only non-trivial element g

$$\mathcal{R}_g^s i\Lambda + \gamma_0 \mathcal{R}_g^{s\,\dagger} i\Lambda\gamma_0 = 2(-)^{ps}\gamma_5 i\Lambda\,. \tag{4.9}$$

As there are no anomalies on odd dimensional spaces when the spin bundle is trivial, it follows that only the trace over the four dimensional part can give rise to an anomaly. Confirming the by now well-know results:

$$\mathcal{A}[\Lambda] = \frac{1}{2}\sum_{s=0,1} 2\pi \int_{\mathbb{R}^{1,3}\otimes S^1/\mathbb{Z}_2} (-)^{ps}\,\Omega_4^1(\Lambda; A, F)\,\delta\left(x_g^\perp - s\,\pi\right). \tag{4.10}$$

Here we have used that the standard gauge anomaly in four dimensions is defined w.r.t. $(1+\gamma_5)/2$ left-handed chirality states, while here we find the trace over γ_5. This cancels one factor of $1/2$ in the general trace formula (4.7) on this orbifold. For the case of anti-periodic fermions on the circle, we see that the anomalies at both fixed points are opposite.

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4.2 10D theories on the orbifold $T^6/\mathbb{Z}_4$

Next we start with ten dimensional Minkowski space $\mathbb{R}^{1,9}$ on which we take partly complex coordinates: $(x, z_1, z_2, z_3) \in \mathbb{R}^{1,3} \times \mathbb{C}^3$. The lattice for this example is taken to be $\Gamma = (0_4, \mathbb{Z}^3 + i\mathbb{Z}^3)$. And the group $G \cong \mathbb{Z}_4$ is generated by the action

$$g(x, z_1, z_2, z_3) = (x, -z_1, i\, z_2, i\, z_3)\,, \qquad g^2(x, z_1, z_2, z_3) = (x, z_1, -z_2, -z_3)\,; \tag{4.11}$$

the element g^2 generates a $\mathbb{Z}_2$ subgroup of G. As in the previous section the conjugacy classes are the elements of the group themselves, since the group is abelian. However, the fixed point structure is more interesting in this case. First of all the projectors defined in (2.1) read in this case

$$\begin{aligned} P^{||}_{1}(x, z_1, z_2, z_3) &= (x, z_1, z_2, z_3)\,, \\ P^{||}_{g}(x, z_1, z_2, z_3) &= (x, 0, 0, 0)\,, \\ P^{||}_{g^2}(x, z_1, z_2, z_3) &= (x, z_1, 0, 0)\,, \end{aligned} \tag{4.12}$$

and $P^{||}_{g^3} = P^{||}_{g}$. Hence we see that the fixed spaces of g and g^3 are isomorphic to multiple copies of four dimensional Minkowski spaces, while the fixed space $\mathbb{T}^{\text{fix}}_{g^2}$ is six dimensional. To describe these fixed point spaces further, we introduce the following definition for the fixed points

$$\mathfrak{z}^{p\,q}_{g} = (0, \zeta_{p_1 q_1}, \zeta_{p_2 p_2}, \zeta_{p_3 p_3})\,, \qquad \mathfrak{z}^{p\,q}_{g^2} = (0, 0, \zeta_{p_2 q_2}, \zeta_{p_3 q_3})\,, \qquad \zeta_{p\,q} = \frac{p + i\, q}{2}\,, \tag{4.13}$$

with $p_j, q_j = 0, 1$, of complex codimension three and two, respectively. The fixed point sets take the form

$$\mathbb{T}^{\text{fix}}_{g} = \sum_{p,q} \mathbb{T}^{\text{fix}}_{g\,p,q}\,, \quad \mathbb{T}^{\text{fix}}_{g\,p,q} = \mathfrak{z}^{p\,q}_{g} + \mathbb{T}^{||}_{g}\,, \qquad \mathbb{T}^{\text{fix}}_{g^2} = \sum_{p,q} \mathbb{T}^{\text{fix}}_{g^2\,p,q}\,, \quad \mathbb{T}^{\text{fix}}_{g^2\,p,q} = \mathfrak{z}^{p\,q}_{g^2} + \mathbb{T}^{||}_{g^2}\,. \tag{4.14}$$

Contrary to the previous example, now the equivalence relation of fixed point labels is not entirely trivial: For $p \neq q$ we find trivially that $g\, \mathfrak{z}^{p,q}_{g^2} = \mathfrak{z}^{q,p}_{g^2}$ up to lattice shifts; hence on the level of the full orbifold the corresponding fixed point spaces are identified. As discussed in subsection 2.1 the precise prescription of this identification is determined by (2.10). The element g acts non trivially on the fixed spaces of g^2, in particular, we find that

$$p^{||}_{g^2}(g)(x, z_1) = (x, -z_1)\,. \tag{4.15}$$

In words this means that the torus T^2 above $\mathfrak{z}^{p,q}_{g^2}$ is identified with the torus T^2 above $\mathfrak{z}^{q,p}_{g^2}$ with opposite orientations. Because of this identification, we define the sum over the indices p, q for the fixed point sets of g^2 up to the interchange of these indices. In particular we will write $p \neq q$ to indicate that the indices are not equal.

With this in mind let us turn to the orbifolding of the fixed point sets. For the fixed points of g, and g^2 with $p \neq q$, the situation is again rather easy, since

$$G^{p,q}_{g} = \text{Ker}^{||}_{g\,p,q} \cong \mathbb{Z}_4\,, \qquad G^{p\neq q}_{g^2} = \text{Ker}^{||}_{g^2\,p\neq q} \cong \mathbb{Z}_2\,. \tag{4.16}$$

(Here we have used that g does not map the fixed space of g^2 with $p \neq q$ to itself.) Hence on the corresponding fixed points there is no residual orbifolding. For the fixed points of g^2 with $p = q$ the situation is different since there

$$G_{g^2}^{p=q} = \langle g \rangle \cong \mathbb{Z}_4 \,, \qquad \mathrm{Ker}^{||}_{g^2\,p=q} = \langle g^2 \rangle \cong \mathbb{Z}_2 \,. \tag{4.17}$$

This implies that $H_{g^2}^{p=q} \cong \mathbb{Z}_4/\mathbb{Z}_2 \cong \mathbb{Z}_2$, and hence the orbifold twist on those fixed points leads to the orbifold $T^2/\mathbb{Z}_2$ above the fixed points $\mathfrak{z}_{g^2}^{p=q}$. Before we can apply the general formula for the orbifold trace (3.7), we need to determine the matrices $((1-h)^{-1})_h^{\perp}$ and the determinants $\det{}_h^{\perp}(1-h)$ for $h = g, g^2$. Remembering that we use here a complex basis while $\det{}_h^{\perp}(1-h)$ is the real determinant, they read

$$\begin{aligned} ((1-g)^{-1})_g^{\perp} &= \frac{1}{2}\,\mathrm{diag}(1, 1+i, 1+i), & ((1-g^2)^{-1})_{g^2}^{\perp} &= \frac{1}{2}\,\mathrm{diag}(1,1), \\ \det{}_g^{\perp}(1-g) &= |2|^2|1-i|^2|1-i|^2 = 16, & \det{}_{g^2}^{\perp}(1-g) &= |2|^2|2|^2 = 16. \end{aligned} \tag{4.18}$$

On a given field ψ the torus and orbifold boundary conditions read

$$\psi(x + e_j) = T_j\, \psi(x) \,, \qquad \psi(x + i\, e_j) = T_{j'}\, \psi(x) \,, \qquad \psi(g\, x) = R\, \psi(x) \,, \tag{4.19}$$

where for $j = 1, 2, 3$ the vectors e_j denote the basis elements for the complex three torus T^6. The consistency of the orbifolding conditions with the periodicity conditions leads to requirements on the matrices $T_j, T_{j'}, R$. For $j = 2, 3$ they take the form

$$T_{j'} = R T_j R^{-1} \,, \qquad R^4 = (T_j R)^4 = (R^2 T_j)^2 = 1 \,. \tag{4.20}$$

This shows that the translations $T_{j'}$ and T_j are not independent. On the other hand, as the g act as a $\mathbb{Z}_2$ action on z_1 the matrices T_1 and $T_{1'}$ are independent, and satisfy

$$(RT_1)^2 = 1 \,, \qquad (RT_{1'})^2 = 1 \,. \tag{4.21}$$

Taking all this information into account, we can write down the expressions for the trace of an arbitrary operator $\mathcal{O}$ which is compatible with this orbifold. The result can be stated as

$$\begin{aligned} \mathrm{Tr}_{\mathbb{T}/\mathbb{Z}_4, T, R}\Big[\mathcal{O}\Big] = \; & \frac{1}{4}\,\mathrm{Tr}_{\mathbb{T}/\mathbb{Z}_4}\Big[\mathcal{O}_1\Big] + \frac{1}{4}\cdot\frac{1}{16}\sum_{p,q}\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{g\,p,q}}\Big[R_g^{p,q}\mathcal{O}_g\Big] + \\ & + \frac{1}{4}\cdot\frac{1}{16}\cdot\sum_{p\neq q}\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{g^2\,p,q}}\Big[R_{g^2}^{p,q}\mathcal{O}_{g^2}\Big] + \frac{1}{4}\cdot\frac{1}{16}\sum_{p=q}\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{g^2\,p,q}/\mathbb{Z}_2}\Big[R_{g^2}^{p,q}\mathcal{O}_{g^2}\Big] + \\ & + \frac{1}{4}\cdot\frac{1}{16}\sum_{p,q}\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{g^3\,p,q}}\Big[R_{g^3}^{p,q}\mathcal{O}_{g^3}\Big] \,. \end{aligned} \tag{4.22}$$

Here the sums are only over the inequivalent fixed points. In this formula we have repressed the explicit reference $T_g^{||}$. The operators $\mathcal{O}_{g^i}$ are given by

$$\mathcal{O}_g = \mathcal{O}\left(x, z; \frac{1}{2}\partial_1, \frac{1+i}{2}\partial_2, \frac{1+i}{2}\partial_3\right),$$

$$\mathcal{O}_{g^2} = \mathcal{O}\left(x, z; \partial_1, \frac{1}{2}\partial_2, \frac{1}{2}\partial_3\right),$$
$$\mathcal{O}_{g^3} = \mathcal{O}\left(x, z; \frac{1}{2}\partial_1, \frac{1-i}{2}\partial_2, \frac{1-i}{2}\partial_3\right), \tag{4.23}$$

and of course $\mathcal{O}_1 = \mathcal{O}$. In writing the derivatives we have again employed complex notation. Notice that if $\mathcal{O}$ is an operator which only depends on the laplacian, $\mathcal{O}_{g^3} = \mathcal{O}_g$.

In this expression only the operators $R_g^{p,q}$, $R_{g^2}^{p,q}$ and $R_{g^3}^{p,q}$ are left to be determined using their definitions (2.21). To this end we first calculate relevant shifts v_g^s using (2.4)

$$\begin{aligned} v_g^{p,q} &= (0,-1,i,i) * \mathfrak{z}_g^{p,q} - \mathfrak{z}_g^{p,q} = (0, -p_1 - iq_1, -p_2, -p_3)\,, \\ v_{g^2}^{p,q} &= (0,0,-1,-1) * \mathfrak{z}_{g^2}^{p,q} - \mathfrak{z}_{g^2}^{p,q} = (0, 0, -p_2 - iq_2, -p_3 - iq_3)\,, \\ v_{g^3}^{p,q} &= (0,-1,-i,-i) * \mathfrak{z}_g^{p,q} - \mathfrak{z}_g^{p,q} = (0, -p_1 - iq_1, -ip_2, -ip_3)\,. \end{aligned} \tag{4.24}$$

Hence the matrices

$$\begin{aligned} R_g^{p,q} &= {T_{v_g^{p,q}}}^{-1} R_g = T_{(0,p_1+iq_1,p_2,p_3)}\, R = T_1^{p_1}\, T_{1'}^{q_1}\, T_2^{p_2}\, T_3^{p_3}\, R\,, \\ R_{g^2}^{p,q} &= {T_{v_{g^2}^{p,q}}}^{-1} R_{g^2} = T_{(0,0,p_2+iq_2,p_3+iq_3)}\, R = T_2^{p_2}\, R T_2^{q_2} R^{-1}\, T_3^{p_3}\, R T_3^{q_3} R^{-1}\, R^2\,, \\ R_{g^3}^{p,q} &= {T_{v_{g^3}^{p,q}}}^{-1} R_{g^3} = T_{(0,p_1+iq_1,ip_2,ip_3)}\, R = T_1^{p_1}\, T_{1'}^{q_1}\, R T_2^{p_2}\, T_3^{p_3} R^{-1}\, R^3\,, \end{aligned} \tag{4.25}$$

encode all the possible effects of non-trivial periodicity boundary conditions on T^6 that are compatible with the $\mathbb{Z}_4$ orbifolding. (Of course, as soon as we take the field ψ to be periodic, i.e. $T_j = T_{j'} = 1$, the traces at all fixed points within a given class are identical: $R_{g^k}^{p,q} = R^k$.) In a future publication [69] we will apply these results to anomalies in heterotic $\mathrm{E}_8 \times \mathrm{E}_8{}'$ theory compactified on this orbifold $T^6/\mathbb{Z}_4$.

4.3 The orbifold T^4/D_4

In our final example we consider a non-abelian orbifold, with the finite group G isomorphic to the dihedral group D_4 with eight elements. The space $\mathbb{T} = \mathbb{R}^{1,7}/\Gamma$ is defined by the four dimensional lattice, which is a complex basis takes the form $\Gamma = \mathbb{Z}^2 + i\,\mathbb{Z}^2$. On $\mathbb{T}$ the action of $G = \langle g, k\rangle \cong D_4$ is defined by the following generating elements

$$g = \begin{pmatrix} \mathbb{1}_4 & & \\ & i & \\ & & -i \end{pmatrix}, \quad k = \begin{pmatrix} \mathbb{1}_4 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \qquad g^4 = k^2 = 1\,, \quad kgk^{-1} = g^{-1}\,. \tag{4.26}$$

Here we also gave their defining properties. The conjugacy classes (h) of $h \in G$ are given by

$$(1) = \{1\}\,, \quad (g) = \{g, g^3\}\,, \quad (g^2) = \{g^2\}\,, \quad (k) = \{k, g^2 k\}\,, \quad (gk) = \{gk, g^3 k\}\,. \tag{4.27}$$

As representatives of these conjugacy classes we take the elements in between the round brackets. As in the previous two examples we need to investigate the properties of these representatives and derived objects. Since the analysis in this case is rather lengthy, we have simply summarized the results in table 1.

h	1	g	g^2	k	gk
$P^{\parallel}_h$	$\begin{pmatrix} \mathbb{1}_4 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} \mathbb{1}_4 & & \\ & 0 & \\ & & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbb{1}_4 & & \\ & 0 & \\ & & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 2\mathbb{1}_4 & & \\ & 1 & 1 \\ & 1 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 2\mathbb{1}_4 & & \\ & 1 & i \\ & -i & 1 \end{pmatrix}$
$d^{\parallel}_h$	8	4	4	6	6
$((1-h)^{-1})^{\perp}_h$	$-$	$\frac{1}{2}\begin{pmatrix} 1+i & \\ & 1-i \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\frac{1}{2}P^{\perp}_k$	$\frac{1}{2}P^{\perp}_{gk}$
$\det^{\perp}_h(1-h)$	1	4	16	4	4
$C(h)$	$\langle g, k\rangle$	$\langle g\rangle$	$\langle g, k\rangle$	$\langle g^2, k\rangle$	$\langle g^2\rangle$
$\mathrm{Ker}^{\parallel}_h$	$\{1\}$	$\langle g\rangle$	$\langle g, k\rangle$	$\langle k\rangle$	$\{1\}$
Fixed points	$\mathfrak{z}_1$ 0	$\mathfrak{z}_{g\,p}$ $(0_4, \zeta_{p_1 p_1}, \zeta_{p_2, p_2})$	$\mathfrak{z}_{g^2\,p,q}$ $(0_4, \zeta_{p_1 q_1}, \zeta_{p_2, q_2})$	$\mathfrak{z}_{k\,p,q}$ $\zeta_{p\,q}(0_4, 1, -1)$	$\mathfrak{z}_{gk\,p,q}$ $\zeta_{p\,q}(0_4, 1, i)$
Shifts v^s_g	$(0_4, 0, 0)$	$(0_4, -p_1, -ip_2)$	$(0_4, -p_1-iq_1, -p_2-iq_2)$	$(0_4, -p-iq, p+iq)$	$(0_4, -p-iq, -ip+q)$

Table 1: The properties of the representatives h of the conjugacy classes (h) in G have been collected in this table. The symbol $\zeta_{p\,q}$ is defined as in eq. (4.13).

Not all the fixed points of h given in table 1 are independent, when we take the identification due to the centralizer $C(h)$ of h into account. The independent fixed points of the various representatives of the conjugacy classes are

$$
\begin{aligned}
\mathfrak{z}_{g\,p}:\ & \mathfrak{z}_{g\,00},\ \mathfrak{z}_{g\,11},\ \mathfrak{z}_{g\,01}\,; \quad \mathfrak{z}_{g^2\,pq}:\ \mathfrak{z}_{00,00},\ \mathfrak{z}_{11,11},\ \mathfrak{z}_{00,11},\ \mathfrak{z}_{01,01},\ \mathfrak{z}_{01,10},\ \mathfrak{z}_{00,01},\ \mathfrak{z}_{01,11}\,; \\
\mathfrak{z}_{k\,pq}:\ & \mathfrak{z}_{k\,00},\ \mathfrak{z}_{k\,11},\ \mathfrak{z}_{k\,01},\ \mathfrak{z}_{k\,10}\,; \quad \mathfrak{z}_{gk\,pq}:\ \mathfrak{z}_{gk\,00},\ \mathfrak{z}_{gk\,11},\ \mathfrak{z}_{gk\,01},\ \mathfrak{z}_{gk\,10}\,.
\end{aligned} \tag{4.28}
$$

Let ψ be a field on this orbifold satisfying the following generating boundary conditions

$$
\begin{aligned}
\psi(x+e_j) &= T_j\,\psi(x), & \psi(x+i\,e_j) &= T_{j'}\,\psi(x)\,, \\
\psi(g\,x) &= R\,\psi(x)\,, & \psi(k\,x) &= S\,\psi(x)\,,
\end{aligned} \tag{4.29}
$$

where e_1, e_2 denote the basis for the complex two torus T^4. The various consistency requirements of these transformations lead to constraints

$$
R^4 = S^2 = 1\,, \qquad SRS^{-1} = R^{-1}\,, \qquad (RT_1)^4 = (R^2T_1)^2 = 1 \tag{4.30}
$$

on T_1, R and S. The other translation elements are not independent, but are determined by

$$
T_{1'} = R\,T_1\,R^{-1}\,, \quad T_2 = S\,T_1\,S^{-1}\,, \quad T_{2'} = SR\,T_1\,R^{-1}S^{-1}\,. \tag{4.31}
$$

(It should be observed that these expressions are not unique.)

Using the indices of the fixed points, defined in (4.28), in the sums below, the trace of a general operator $\mathcal{O}$ on this orbifold becomes

$$
\begin{aligned}
\mathrm{Tr}_{\mathbb{T}/G,T,R}\Big[\mathcal{O}\Big] = {} & \frac{1}{8}\,\mathrm{Tr}_{\mathbb{T}/G,T,R}\Big[\mathcal{O}\Big] + \frac{1}{8}\cdot\frac{1}{4}\sum_{p}\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{g\,p},T^{\parallel}_g}\Big[R^p_g\mathcal{O}_g\Big] + \\
& + \frac{1}{8}\cdot\frac{1}{16}\sum_{p,q}\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{g^2\,p,q},T^{\parallel}_{g^2}}\Big[R^{p,q}_{g^2}\mathcal{O}_{g^2}\Big] + \frac{1}{8}\cdot\frac{1}{4}\sum_{p,q}\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{k\,p,q}/\langle g^2\rangle,T^{\parallel}_k}\Big[R^{p,q}_k\mathcal{O}_k\Big] +
\end{aligned}
$$

$$+\frac{1}{8}\cdot\frac{1}{4}\sum_{p,q}\mathrm{Tr}_{\mathbb{T}^{\mathrm{fix}}_{gkp,q},T^{\|}_{gk}}\left[R^{p,q}_{gk}\mathcal{O}_{gk}\right] \tag{4.32}$$

where $\mathcal{O}_1 = \mathcal{O}$ and

$$\begin{aligned} \mathcal{O}_g &= \mathcal{O}\left(x,z;\partial_x,\frac{1+i}{2}\partial_{z_1},\frac{1-i}{2}\partial_{z_2}\right), & \mathcal{O}_{g^2} &= \mathcal{O}\left(x,z;\partial_x,\frac{1}{2}\partial_{z_1},\frac{1}{2}\partial_{z_2}\right), \\ \mathcal{O}_k &= \mathcal{O}\left(x,z;\partial_x,\partial^{\|}_k,\frac{1}{2}\partial^{\perp}_k\right), & \mathcal{O}_k &= \mathcal{O}\left(x,z;\partial_x,\partial^{\|}_{gk},\frac{1}{2}\partial^{\perp}_{gk}\right). \end{aligned} \tag{4.33}$$

And finally the matrices that appear within the traces at the various fixed hyper surfaces take the form

$$\begin{aligned} R^p_g &= T_{(0_4,p_1,ip_2)}\,R = T_1^{p_1}\,T_{2'}^{p_2}\,R\,, \\ R^{p,q}_{g^2} &= T_{(0_4,p_1+iq_1,p_2+iq_2)}\,R^2 = T_1^{p_1}\,T_{1'}^{q_1}\,T_2^{p_2}\,T_{2'}^{q_2}\,R^2\,, \\ R^{p,q}_k &= T_{(0_4,p+iq,-p-iq)}\,S = T_1^p\,T_{1'}^q\,T_2^{-p}\,T_{2'}^{-q}\,S\,, \\ R^{p,q}_{gk} &= T_{(0_4,p+iq,ip-q)}\,RS = T_1^p\,T_{1'}^q\,T_2^{-q}\,T_{2'}^p\,RS\,. \end{aligned} \tag{4.34}$$

Hence we see that even though this is a more complicated orbifold than the abelian ones, with the methods described in this work they can be analyzed in the same fashion, even if we allow for field which are non-periodic on the torus.

5. Conclusions

We have discussed a general procedure to compute all kinds of one-loop amplitudes on a fairly general class of orbifolds. It is well-known that quantum corrections can be very important in four dimensions; and even more so in extra dimensions, because these theories are non-renormalizable. Models based on orbifolds are very popular to perform investigations of the possible (quantum) physics in extra dimensions. However, most recent field theory investigations have been performed with rather simple orbifolds, like $S^1/\mathbb{Z}_2$. Therefore, it may not be straightforwardly to extended such calculations to more extra dimensions or more complicated orbifold groups. The orbifold group may have subgroups or be non-abelian. For this reason it is important to have some general methods available to perform such field theory calculations on orbifolds. In this paper we have described the tools, which can be used to perform such computations. Our main results may be summarized as follows:

One-loop amplitudes on a general class of orbifolds can be expressed as a sum of one-loop amplitudes, with trivial orbifolding. Therefore, these amplitudes are not more difficult to compute than those on (lower dimensional) tori. As our results only require that the corresponding operator is compatible with the orbifolding (i.e. it satisfies condition (2.29)) our results can be applied to a wide range of possible computations on orbifolds. We have illustrated the general methods exposed in this paper by consider the examples of the orbifolds: $S^1/\mathbb{Z}_2$, $T^6/\mathbb{Z}_4$ and T^4/D_4. The first example has been often studied in the past, and provides the reader with an easy test case of our general methods, while the other two examples show that they can equally well be applied to non-prime and even non-abelian orbifolds. Moreover, the fields are allowed to have twisted periodicity conditions around non-contractible cycles of the torus. This leads to different "projections" at the various fixed points which are then distinguished by the Wilson lines.

In more technical detail our results amount to the following: Any one-loop calculation can be formulated as the evaluation of a given operator on the Hilbert space of states, which run around in the loop. The properties of this Hilbert space are determined by the characteristics of the vector bundle in which those states take their values. This more abstract point of view has the advantage, that it allows us to treat orbifold calculations for many different types of fields, corresponding to various bundles, in an integrated way. The flat orbifolds, considered in this work, are obtained by taking d dimensional Minkowski spaces divided by an n dimensional lattice Γ, and subsequently by a finite group $G \subset \mathrm{SO}(1, \mathrm{d}-1)$. We have denoted this "torus" as $\mathbb{T} = \mathbb{R}^{1,d-1}/\Gamma$ and the resulting orbifold by $\mathbb{T}/G$. The orbifold group G is often taken to be abelian. However, our results apply to abelian and non-abelian orbifolds alike. The properties of fields on this spaces are determined by their transformation properties under lattice shifts and the action of the elements of the finite group G. The fields need not be periodic under these lattices shifts, nor do they have to transform trivially under G. Therefore, the bundle in which the fields live, are described by two homomorphisms T and R, encoding the non-trivial periodicities and orbifold twists, respectively. In this work we have shown how the traces over the Hilbert spaces twisted by R can be expressed as a sum of traces over Hilbert spaces which are not twisted by the orbifolding. These Hilbert spaces may correspond to bundles over lower dimensional tori or orbifolds, and involve local projections $R^{s_r}_{g_r}$. These local projectors take the effects of non-trivial Wilson lines into account. In addition, we found that each of these subtraces come with a normalization factor $1/\det{}^{\perp}_{g}(1-g)$, and the derivatives on the delta function are rescaled, according to $\partial^{\perp}_{g} \rightarrow (\partial(1-g)^{-1})^{\perp}_{g}$. We have given a general prescription of how the orbifolded fixed point spaces can be identified.

Let us close the conclusion by giving some examples to which our results can be applied. In this work we have mentioned only a few applications: the computation of anomalies, tadpoles and vacuum energies on general orbifolds. However, as the results can be applied to any operator on an orbifold, it can also be used to compute the renormalization of the kinetic terms or the (gauge) couplings, both in the bulk, as well as, on the orbifold fixed hyper surfaces. In this work we have restricted ourselves to flat global orbifolds, i.e. the covering space has been taken the flat d dimensional Minkowski space $\mathbb{R}^{1,d-1}$. We would expect that the methods can be extended to the case where the covering space is a general curved manifold, that possesses a certain amount of isometries. Also, it should be possible to formulate the prescriptions such that they can be applied to local orbifolds, where the orbifolding group for the various coordinate patches may be different.

Acknowledgments

The author would like to thank T. Kobayashi and M.G.A. Walter, for inspiring discussions concerning heterotic T^6/Z_4 orbifolds that have initiated this project, and M. Olechowski and M. Vonk for some useful communications. The author also acknowledges the kind hospitality of the group of H.P. Nilles in Bonn where this work was completed.

This work has been supported by NSERC and CITA.

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A. Complex scalar functions on $\mathbb{T}$

Complex scalar functions define the simplest examples of sections of bundles over $\mathbb{T}$ or $\mathbb{T}/G$. Most other more complicated bundles can be obtained by taking a tensor product of this complex line bundle times other bundles. In particular, if one wants to give an explicit basis for the Hilbert space associated with a given bundle, then one needs the basis functions of the scalar Hilbert space which we now describe. The dual or momentum space $\mathbb{T}^*$ of $\mathbb{T}$ is defined as

$$\mathbb{T}^* = \Big\{ q \in (\mathbb{R}^{1,d-1})^* \,|\, \forall v \in \Gamma : \; q^T \eta \, v \in 2\pi \, \mathbb{Z} \Big\} . \tag{A.1}$$

The elements of $\mathbb{T}^*$ can be used to parameterize a basis of the Hilbert space of complex scalar functions on $\mathbb{T}$:

$$\phi(x;q) = \frac{e^{i\, q^T \eta \, x}}{\sqrt{N}} , \qquad \partial_\mu \phi(x;q) = i\, (q^T \eta)_\mu \phi(x;q) , \qquad \forall v \in \Gamma : \; \phi(x+v;q) = \phi(x;g) . \tag{A.2}$$

which are periodic w.r.t. Γ. The normalization $N = (2\pi)^{d-n} \mathrm{Vol}_\Gamma$ takes into account the usual factors of 2π, that arise with Fourier transforms, and the volume Vol_Γ of a fundamental domain of the lattice Γ. Their orthonormality relations read

$$\int_{\mathbb{T}} \mathrm{d}x \, \phi(x;q) \phi(x;q')^* = \delta_{\mathbb{T}^*}(q-q') , \qquad \int_{\mathbb{T}^*} \mathrm{d}q \, \phi(x;q) \phi(x';q)^* = \delta_{\mathbb{T}}(x-x') , \tag{A.3}$$

Here we have employed Lebeque integration to avoid having to write sums and integrals over the discrete and continuous parts of $\mathbb{T}^*$ separately. With $\delta_{\mathbb{T}^*}(p)$ we denote the Kronnecker delta on the discrete part of $\mathbb{T}^*$ and the ordinary delta on the remaining $d-n$ directions in $\mathbb{T}^*$. In section 2.1 we defined the projection operators $P_g^{\|,\perp}$ of an element $g \in G$. Because of the direct product structure (2.2) of $\mathbb{T}$, it follows that the mode functions (A.2) factorize as

$$\phi(x;q) = \phi_g^{\|}(x;q) \, \phi_g^{\perp}(x;q) , \tag{A.4}$$

Here $\phi_g^{\|}(x;q) = \sqrt{N_g^{\perp}} \phi(x; P_g^{\|} q)$ and $\phi_g^{\perp}(x;q) = \sqrt{N_g^{\|}} \phi(x; P_g^{\perp} q)$ are complete sets of mode functions on $\mathbb{T}_g^{\|}$ and $\mathbb{T}_g^{\perp}$, respectively. Here for $i = \|, \perp$ we have defined $N_g^i = (2\pi)^{d_g^i - n_g^i} \mathrm{Vol}_{\Gamma_g^i}$, in terms of the dimension n_g^i of the lattice Γ_g^i, and its volume $\mathrm{Vol}_{\Gamma_g^i}$. Hence, we have similar orthonormality relations as (A.3) on $\mathbb{T}_g^{\|,\perp}$

$$\int_{\mathbb{T}_g^i} \mathrm{d}x \, \phi_g^i(x;q) \phi_g^i(x;q')^* = \delta_g^{i\,*}(q-q'), \qquad \int_{\mathbb{T}_g^{i\,*}} \mathrm{d}q \, \phi_g^i(x;q) \phi_g^i(x';q)^* = \delta_g^i(x-x'). \tag{A.5}$$

In addition, a useful property of the perpendicular scalar mode functions is

$$\phi_g^{\perp}(gx;q)^* \partial_x \, \phi_g^{\perp}(x;q) = \big[\partial_x (1-g)^{-1} \big]^{\perp}_{g} \phi_g^{\perp}(x;q) \phi_g^{\perp}(gx;q)^* . \tag{A.6}$$

Since by assumption the fiber $\mathbb{F}$ is a complex vector space, we can use these scalar mode functions to define a basis for the Hilbert space $\mathcal{H}_{\mathbb{T}}$. Let ϵ_σ be a basis for the vector space $\mathbb{F}$, i.e.

$$\epsilon_\sigma{}^\dagger \epsilon_{\sigma'} = \delta_{\sigma\,\sigma'} \qquad \sum_\sigma \epsilon_\sigma \epsilon_\sigma^\dagger = \mathbb{1} . \tag{A.7}$$

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An orthonormal basis $|\phi_\sigma(q)\rangle$ for the Hilbert space $\mathcal{H}_\mathbb{T}$ is given in the coordinate space by $\phi_\sigma(x;q) = \phi(x;q)\,\epsilon_\sigma$:

$$\langle\phi_\sigma(q)\,|\,\phi_{\sigma'}(q')\rangle = \int_\mathbb{T} \mathrm{d}x\,\phi_\sigma(x;q)^\dagger\phi_{\sigma'}(x;q') = \delta_{\sigma\,\sigma'}\delta_{\mathbb{T}^*}(q-q')\,. \tag{A.8}$$

B. Orbifold Hilbert space trace computations

This appendix is devoted to the derivation of the central results given in section 3.1. In particular we prove the identities (3.4) and (3.5).

To obtain equation (3.4), write out the trace in the orthonormal basis (A.8) of the Hilbert space $\mathcal{H}_{\mathbb{T},T}$ and the definition of the projection operator (2.25), and subsequently employ the change of variables: $y = h^{-1}x$ and $h^{-1}g \to g$:

$$\begin{aligned}\mathrm{Tr}_{\mathbb{T},T,R}[\mathcal{O}] &= \frac{1}{|G|^2}\sum_{g,h\in G}\sum_\sigma \int_{\mathbb{T}\otimes\mathbb{T}^*}\mathrm{d}x\,\mathrm{d}q\;\phi_\sigma(g^{-1}x;q)^\dagger\mathcal{T}(g^{-1}x)^{-1}\times\\ &\qquad\qquad\times R_g^\dagger\,\rho\,\mathcal{O}(x,\partial)\,R_h\mathcal{T}(h^{-1}x)\phi_\sigma(h^{-1}x;q)\\ &= \frac{1}{|G|^2}\sum_{g,h\in G}\sum_\sigma \int_{\mathbb{T}\otimes\mathbb{T}^*}\mathrm{d}y\,\mathrm{d}q\;\phi_\sigma(g^{-1}y;q)^\dagger\mathcal{T}(g^{-1}y)^{-1}\times\\ &\qquad\qquad\times R_g^\dagger\,\rho\;R_h^{-1}\mathcal{O}(hy,\partial_y h^{-1})R_h\,\mathcal{T}(y)\phi_\sigma(y;q)\,.\end{aligned}\tag{B.1}$$

This expression is the explicit form of equation (3.4).

Next we compute the $\mathrm{Tr}_{\mathbb{T},T}[\mathcal{P}_R{}^\dagger\mathcal{O}]$ for an arbitrary operator $\mathcal{O}$ on $\mathcal{H}_{\mathbb{T},T}$ to confirm equation (3.5). We first employ the splitting of the basis of the Hilbert space into parallel and perpendicular components w.r.t. each $g \in G$

$$\begin{aligned}\mathrm{Tr}_{\mathbb{T},T}\Big[\mathcal{P}_R{}^\dagger\mathcal{O}\Big] &= \frac{1}{|G|}\sum_{g\in G}\sum_\sigma \int_{\mathbb{T}\otimes\mathbb{T}^*}\mathrm{d}x\,\mathrm{d}q\;\phi^{\parallel}_{g\,\sigma}(x;q)^\dagger\phi^\perp_g(g^{-1}x;q)^*\mathcal{T}^\parallel_g(x)^{-1}\mathcal{T}^\perp_g(g^{-1}x)^{-1}R_g{}^\dagger\times\\ &\qquad\qquad\times\rho\sum_{\vec{\mu}}\mathcal{O}^{\vec{\mu}}(x)\,\partial_{\vec{\mu}}\,\mathcal{T}^\perp_g(x)\mathcal{T}^\parallel_g(x)\phi^\perp_g(x;q)\phi^\parallel_{g\,\sigma}(x;q)\,.\end{aligned}\tag{B.2}$$

Here we have used that the points in the subspace $\mathbb{T}^\parallel_g$ are inert under the action of g. The next step is to bring all functions of $x^\perp_g$ on the right side of the derivatives. To this end, we move $\mathcal{T}^\perp_g(g^{-1}x)^{-1}$ to the left, and insert another decomposition of unity $\sum_{\sigma'}\epsilon_{\sigma'}\epsilon_{\sigma'}{}^\dagger = 1$ after it. Keeping track on what the derivatives $\partial_{\vec{\mu}}$ act, we can take $\epsilon_\sigma{}^\dagger\mathcal{T}^\perp_g(g^{-1}x)^{-1}\epsilon_{s'}$ and $\phi^\perp_g(g^{-1}x;q)^*$ to the far r.h.s. of this equation. Using the closure $\sum_\sigma\epsilon_\sigma\epsilon_\sigma{}^\dagger = 1$ once again, and the matrix version of (A.6)

$$\begin{aligned}&\partial_{\vec{\mu}}\Big(\phi^\perp_g(x;q)\mathcal{T}^\perp_g(x)\Big)\Big[\mathcal{T}^\perp_g(g^{-1}x)^{-1}\phi^\perp_g(g^{-1}x;q)^*\Big] =\\ &\qquad = \Big[\partial^\parallel_g + (\partial(1-g^{-1})^{-1})^\perp_g\Big]_{\vec{\mu}}\Big(\mathcal{T}^\perp_g(x)\mathcal{T}^\perp_g(g^{-1}x)^{-1}\phi^\perp_g(x;q)\phi^\perp_g(g^{-1}x;q)^*\Big)\,,\end{aligned}$$

we obtain

$$\mathrm{Tr}_{\mathbb{T},T}\Big[\mathcal{P}_R{}^\dagger\mathcal{O}\Big] = \frac{1}{|G|}\sum_{g\in G}\sum_\sigma \int_{\mathbb{T}\otimes\mathbb{T}^{\parallel}_{g\,*}}\mathrm{d}x\,\mathrm{d}q\;\phi^\parallel_{g\,\sigma}(x;q)^\dagger\mathcal{T}^\parallel_g(x)^{-1}R_g{}^\dagger\,\rho\sum_{\vec{\mu}}\mathcal{O}^{\vec{\mu}}(x)\,\mathcal{T}^\perp_g(x)\times$$

$$\times\Big[\partial_g^{||}+(\partial(1-g^{-1})^{-1})_g^{\perp}\Big]_{\vec\mu}\times \tag{B.3}$$
$$\times\Big(\mathcal{T}_g^{||}(x)\mathcal{T}_g^{\perp}(g^{-1}x)^{-1}\phi_{g\sigma}^{||}(x;q)\int_{\mathbb{T}_g^{\perp *}}\mathrm{d}q\phi_g^{\perp}(g^{-1}x;q)^*\phi_g^{\perp}(x;q)\Big).$$

The closure relation (A.5) on $\mathbb{T}_g^{\perp}$ and (2.6) allows us to rewrite this as contributions on the fixed spaces of g:

$$\begin{aligned}\mathrm{Tr}_{\mathbb{T},T}\Big[\mathcal{P}_R{}^{\dagger}\mathcal{O}\Big] &= \frac{1}{|G|}\sum_{g\in G}\frac{1}{|\det_g^{\perp}(1-g^{-1})|}\sum_{s,\sigma}\int_{\mathbb{T}\otimes\mathbb{T}_g^{||*}}\mathrm{d}x\mathrm{d}q\,\phi_{g\sigma}^{||}(x;q)^{\dagger}\mathcal{T}_g^{||}(x)^{-1}R_g{}^{\dagger}\rho\times\\ &\times\sum_{\vec\mu}\mathcal{O}^{\vec\mu}(\mathfrak{z}_g^s+x_g^{||})\Big[\partial_g^{||}+(\partial(1-g^{-1})^{-1})_g^{\perp}\Big]_{\vec\mu}\times\\ &\times\Big(\mathcal{T}_g^{\perp}(\mathfrak{z}_g^s)\mathcal{T}_g^{\perp}(g^{-1}\mathfrak{z}_g^s)^{-1}\mathcal{T}_g^{||}(x)\phi_{g\,\sigma}^{||}(x;q)\delta_g^{\perp}(x-\mathfrak{z}_g^s)\Big)\,.\end{aligned}\tag{B.4}$$

Here we have substituted $x_g^{\perp}=\mathfrak{z}_g^s$ everywhere, using the defining property of the delta function $\delta_g^{\perp}$. It should be noted, that one has to be careful employing this property here, because of the presence of the derivative $\partial_g^{\perp}$. However, since we made sure in the previous step that all $x_g^{\perp}$ dependence is under the derivatives, we may safely use this property.[2] Since now, $\mathcal{T}_g^{\perp}(\mathfrak{z}_g^s)\mathcal{T}_g^{\perp}(g^{-1}\mathfrak{z}_g^s)^{-1}$ is constant, we can pull it outside of the derivatives. Taking $g\to g^{-1}$ and using that the projector of g and g^{-1} are identical (see (2.1)), and employing the reserve method of including complete sets, allows us to rewrite this as

$$\begin{aligned}\mathrm{Tr}_{\mathbb{T},T}\Big[\mathcal{P}_R{}^{\dagger}\mathcal{O}\Big] &= \frac{1}{|G|}\sum_{g\in G}\frac{1}{|\det_g^{\perp}(1-g)|}\sum_{\sigma}\int_{\mathbb{T}_g^{\mathrm{fix}}\otimes\mathbb{T}_g^{||*}}\mathrm{d}x\,\mathrm{d}q\;\phi_{g\,\sigma}^{||}(x;q)^{\dagger}\mathcal{T}_g^{||}(x)^{-1}\mathcal{R}_g(x)\times\\ &\times\sum_{\vec\mu}\mathcal{O}^{\vec\mu}(x)\,\Big[\partial_g^{||}+(\partial(1-g)^{-1})_g^{\perp}\Big]_{\vec\mu}\mathcal{T}_g^{||}(x)\phi_{g\,\sigma}^{||}(x;q)\,,\end{aligned}\tag{B.5}$$

where we have employed (2.21) and (2.24); and thereby we have obtained (3.5).

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